TJUSAMO 2011 – Enumerative Combinatorics

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1 Bijection

Enumeration, in the mathematical sense, is counting how many elements are in a set. Although this can be done by expressing this number as a sum/product and evaluating it, it is almost always more illuminating to count the elements in a set by establishing a bijection. A bijection between two sets A and B is a function f which is both one-to-one and onto – that is, it is a one-to-one correspondence between elements of A and elements of B. If there exists a bijection between A and B, then A and B have the same cardinality.
The canonical example of this method is counting how many solutions there are to $a_1+a_2+\cdots+a_k = n$ in nonnegative integers $a_1, a_2, \cdots, a_k$. This is done by establishing a bijection between solutions to this equation and arrangements of n "stars" and $k-1$ "bars" in a line.

2 Principle of Inclusion-Exclusion

The principle of Inclusion-Exclusion states that if $\mathcal{F}$ is a finite family of finite sets, then

$$|\cup \mathcal{F}| = \sum_{\mathcal{S}\subseteq\mathcal{F},\, \mathcal{S}\neq\emptyset} (-1)^{|\mathcal{S}|+1} |\cap \mathcal{S}|.$$

Here, $\cup\mathcal{F}$ denotes the union of all the sets in $\mathcal{F}$, and $\cap\mathcal{S}$ the intersection of all the sets in $\mathcal{S}$. Written out in full form, it looks like this:

$$\begin{aligned}|A_1 \cup A_2 \cup \cdots \cup A_n| = {} & (|A_1| + |A_2| + \cdots + |A_n|) \\ & - (|A_1 \cap A_2| + |A_1 \cap A_3| + \cdots + |A_{n-1} \cap A_n|) \\ & + (|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n|) \\ & \;\vdots \\ & \pm (|A_1 \cap A_2 \cap \cdots \cap A_n|).\end{aligned}$$

This is often used in proving combinatorial identities. The simpler statements

$$|A_1 \cup A_2 \cup \cdots \cup A_n| \leq |A_1| + |A_2| + \cdots + |A_n|$$

or

$$|A_1 \cup A_2 \cup \cdots \cup A_n| \geq (|A_1| + |A_2| + \cdots + |A_n|) - (|A_1 \cap A_2| + |A_1 \cap A_3| + \cdots + |A_{n-1} \cap A_n|)$$

will often suffice in other contexts.

3 Identities

Combinatorial identities involving sums can be solved by evaluating the sums algebraically. However, it is almost always more illuminating to prove them combinatorially, using the method of "casework via summation".
The canonical example of proving identities combinatorially is found in the evaluation of the sum

$$\sum_{k=0}^{n} \binom{x+k}{k} = \binom{x+n+1}{n}.$$

To prove this, we consider $x+1$-element sets of $\{1, 2, \cdots, x+n+1\}$, using casework on the largest element of the set.

4 The Pigeonhole Principle

Sometimes, once you count what you've needed to, you can simply write down the result (with proof, of course) and move on to the next problem. Far more often, however, you will need to take an extra (sometimes trivial) step. The pigeonhole principle states that if n pigeons are placed into k holes, then one hole must have at least n/k pigeons in it. (that is, it must have at least $\lceil n/k \rceil$ pigeons in it.)
Applying this can be somewhat tricky, because it isn't always obvious what to choose as "holes". For example, see problem below.

5 Probability and Expected Value

Let X be a random variable which takes values in some finite set S. Then the *expected value* of X, denoted $\mathbb{E}[X]$, is

$$\mathbb{E}[X] = \sum_{x \in S} x \, \mathbb{P}[X = x]$$

Also, we have $X \geq \mathbb{E}[X]$ with nonzero probability. The pigeonhole principle is a special case of this.

6 Problems

1. Let S and T be disjoint one-element sets. Find the number of elements of their union $S \cup T$.

2. In how many ways can you select two disjoint subsets from the set $\{1, 2, 3, \ldots, n\}$?

3. There are eight ways to win in a regular game of tic-tac-toe: take one of the three rows, three columns, or two diagonals. In how many ways can one win on a $k \times k \times k \times \cdots \times k$ (n-dimensional) board? Winning means taking a straight line of k hypercubes.

4. Prove that the number of subsets of $\{1, 2, \ldots, n\}$ with an odd number of elements is equal to the number of subsets of $\{1, 2, \ldots, n\}$ with an even number of elements.

5. Two rows of ten pegs are lined up and adjacent pegs are spaced 1 unit apart. How many ways can ten rubber-bands be looped around the pegs so that no peg does not contain a rubber band? (Rubber bands cannot stretch more than $\sqrt{2}$ units.)

6. Prove that the number of subsets of $\{1, 2, \cdots, n\}$ with no two consecutive elements is the Fibonacci number F_{n+1}.

7. A child sits on each seat in a circular row of n seats. Each child may move by at most one seat. Find the number of ways they can rearrange.

8. Prove that the Fibonacci numbers satisfy

$$F_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots$$

9. Interpret the sum

$$\sum_{k=1}^{n} k^3 \binom{n}{k}$$

combinatorially.

10. Show via a bijection that

$$\binom{m+n}{m} = \sum_{i=0}^{n} \binom{k+i}{k} \binom{m+n-k-i-1}{m-k-1}$$

11. Consider all r-element subsets of $\{1, 2, 3, \ldots, n\}$. Each subset has some k^{th}-largest element. What is the arithmetic mean of these k^{th}-largest elements?

12. Given p and q coprime positive integers, determine the value of

$$\left\lfloor \frac{p}{q} \right\rfloor + \left\lfloor \frac{2p}{q} \right\rfloor + \left\lfloor \frac{3p}{q} \right\rfloor + \cdots + \left\lfloor \frac{(q-1)p}{q} \right\rfloor$$

13. Show that the Catalan numbers $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$ count the number of elements of the sets given below:

    (a) triangulations of a convex n-gon
    (b) parenthasizations in a nonassociative product of n factors
    (c) binary trees with n vertices
    (d) plane trees with $n+1$ vertices
    (e) Dyck paths from (0,0) to $(2n, 0)$, i.e., lattice paths with steps (1,1) and $(1, -1)$, never falling below the x-axis
    (f) lattice paths up and to the right from $(0, 0)$ to (n, n), which never go above the first diagonal.
    (g) ways of connecting $2n$ points in the plane lying on a horizontal line by n nonintersecting arcs, each arc connecting two of the points and lying above the points
    (h) sequences $1 \leq a_1 \leq \cdots \leq a_n$ of integers with $a_i \leq i$; for example $\{111, 112, 113, 122, 123\}$
    (i) plane trees with n vertices whose leaves at height one are colored either red or blue
    (j) ways in which can you put the numbers $1, 2, \ldots, 2n$ into a $2 \times n$ grid such that each number is greater than all the numbers to the right and below it

14. Explain the significance of the following sequence:

    un, dos, tres, quatre, cinc, sis, set, vuit, nou, deu, ...

15. Let S be a finite set, and let $A_1, A_2, \cdots, A_{4^{n-1}}$ be subsets of size n of S. Prove that the elements of S can be colored in four colors such that none of the sets A_i is monochromatic (that is, contains elements of only one color).

16. Some checkers placed on an $n \times n$ checkerboard satisfy the following conditions:

    (1) every square that does not contain a checker shares a side with one that does;
    (2) given any pair of squares that contain checkers, there is a sequence of squares containing checkers, starting and ending with the given squares, such that every two consecutive squares of the sequence share a side.

    Prove that at least $\dfrac{n^2-2}{3}$ checkers have been placed on the board.

17. Let n be a positive integer. Each point (x, y) in the plane, where x and y are non-negative integers with $x + y < n$, is colored red or blue, subject to the following condition: if a point (x, y) is red, then so are all points (x', y') with $x' \leq x$ and $y' \leq y$. Let A be the number of ways to choose n blue points with distinct x-coordinates, and let B be the number of ways to choose n blue points with distinct y-coordinates. Prove that $A = B$.

18. You have n square envelopes of different sizes. You may place smaller envelopes inside larger envelopes, and you may place an unlimited number of envelopes inside any one envelope (as long as the envelopes are smaller than their container).

    (a) How many ways can you arrange your n envelopes?

    (b) Let $\left\langle {n \atop k} \right\rangle$ be the number of ways in which you can arrange your n envelopes if exactly k envelopes do not contain any other envelope. Find a recurrence for $\left\langle {n \atop k} \right\rangle$ not involving sums.

19. A square $(n-1) \times (n-1)$ is divided into $(n-1)^2$ unit squares in the usual manner. Each of the n^2 vertices of these squares is to be colored red or blue. Find the number of different colorings such that each unit square has exactly two red vertices. (Two coloring schemes are regarded as different if at least one vertex is colored differently in the two schemes.)

20. On a given 2008×2008 chessboard, every unit square is filled with one of the letters C, G, M, O. The resulting board is called harmonic if every 2×2 subsquare contains all four different letters. How many harmonic boards are there?

21. Let $\mathcal{F}$ be a set of subsets of the set $\{1, 2, \cdots, n\}$ such that

    (1) if A is an element of $\mathcal{F}$, then A contains exactly three elements;

    (2) if A and B are two distinct elements in $\mathcal{F}$, A and B share at most one common element.

    Let $f(n)$ denote the maximum number of elements in $\mathcal{F}$. Prove that

$$\frac{(n-1)(n-2)}{6} \leq f(n) \leq \frac{(n-1)n}{6}.$$

22. Let S be a finite set with subsets $A_1, A_2, \cdots, A_k$ for which A_i is not a subset of A_j for any $i \neq j$. Prove that

$$\sum_{i=1}^{k} \frac{1}{\binom{|S|}{|A_i|}} \leq 1.$$

23. In a contest, there are m candidates and n judges, where $n \geq 3$ is an odd integer. Each candidate is evaluated by each judge as either pass or fail. Suppose that each pair of judges agrees on at most k candidates. Prove that

$$\frac{k}{m} \geq \frac{n-1}{2n}.$$

24. Prove that $\displaystyle\sum_{i=0}^{n} (-1)^{n+i} \binom{in}{n} \binom{n}{i} = n^n$ for all positive integers n.

25. Prove that
$$\sum_{k=0}^{n} \binom{n}{k}\binom{k}{\lfloor \frac{k}{2} \rfloor}\binom{n-k}{\lfloor \frac{n-k}{2} \rfloor} = \binom{n}{\lfloor \frac{n}{2} \rfloor}\binom{n+1}{\lfloor \frac{n+1}{2} \rfloor}$$

26. Let p be a prime and let a, b be positive integers.

    (a) Prove that $\binom{a}{b} \equiv \binom{ap}{bp} \pmod{p}$.

    (b) Prove that $\binom{a}{b} \equiv \binom{ap}{bp} \pmod{p^2}$.

    (c) Suppose $p > 3$. Prove that $\binom{a}{b} \equiv \binom{ap}{bp} \pmod{p^3}$.

27. Define the sequence a_n by $\sum_{d|n} a_d = 2^n$. Prove that $n|a_n$.

28. Prove that any integer k greater than 1 has a multiple less than k^4 which has at most four distinct digits.

29. Given positive integers d and n with d being a divisor of n. Set S contains all n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $0 \leq x_1 \leq \cdots \leq x_n \leq n$ and $d|x_1 + \cdots + x_n$. Prove that exactly half of the elements in S satisfy the property $x_n = n$.

30. In an international tennis tournament, n players played against each other participant exactly once. There were no ties; every round had a winner and a loser. Suppose there exist two disjoint sets A and B of X players each such that every player from set A defeated every player from set B. Determine the maximum possible X for that statement to remain true, no matter the results of the tournament.

31. Let n be a positive integer. A sequence of n positive integers (not necessarily distinct) is called *full* if it satisfies the following condition: for each positive integer $k \geq 2$, if the number k appears in the sequence then so does the number $k-1$, and moreover the first occurrence of $k-1$ comes before the last occurrence of k. For each n, how many full sequences are there?

32. Prove that the number of trees with n vertices (labeled $\{1, 2, \cdots, n\}$) is n^{n-2}.