

AN INTRODUCTION TO DERIVATIVES

Part 1: Rates of Change

The idea of a rate of change is found in many fields: population growth, heart rate, birth and death rates, unemployment rate, radioactive decay, production rates, marginal revenue, fluid flow, current, velocity, acceleration, etc. In all of these studies, one variable can be expressed as a function of another, i.e., as a function, $y = f(x)$. We will study the change of y as x changes.

$$\text{Rate of change of } y \text{ with respect to } x = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}$$

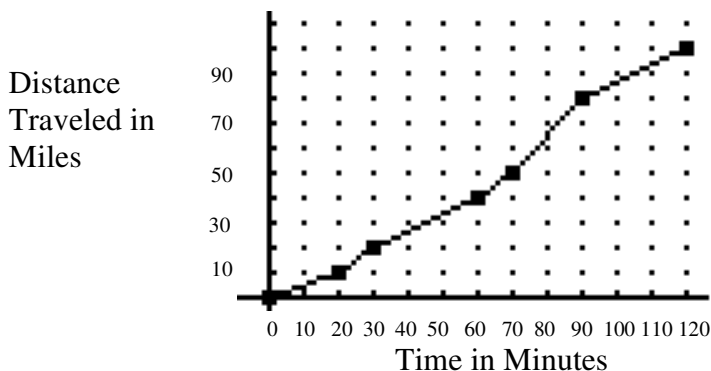
If x changes from a to b , then y changes from $f(a)$ to $f(b)$.
$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

Example 1: After a person has stopped drinking, their blood alcohol level decreases slowly as time passes. Let $C(t)$ represent the blood alcohol concentration. Write an expression for the rate of change of the person's blood alcohol level concentration over the period from $t = 1$ hour to $t = 3$ hours after the person has stopped drinking.

Solution:
$$\frac{\Delta C}{\Delta t} = \frac{C(3) - C(1)}{3 - 1} = \frac{C(3) - C(1)}{2}$$

The study of rate of change is fundamental to the study of calculus. Rates of change can be represented graphically.

Example 2: Suppose the following graph is used to plot the distance a car travels over a two hour period.



What is the average rate the car travels? This question is easy to answer. We can go back to an Algebra 1 equation,

distance = rate • time and solve for the rate; $r = \frac{d}{t}$. Over a two-hour period the car has an average rate of

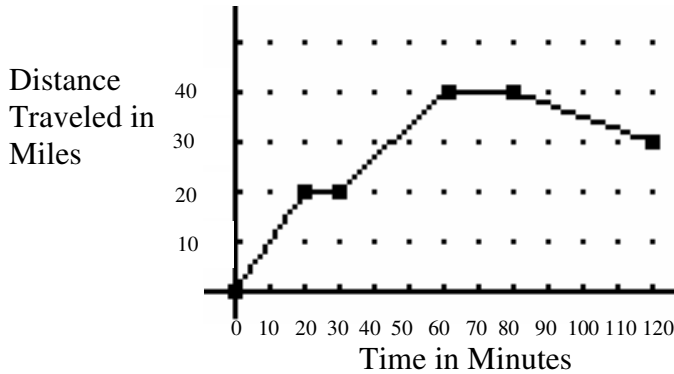
100 mi./2 hr. or 50 mph. Study the graph and answer the following questions. (Be sure to change minutes into hours.)

- How fast is the car going during the second hour?
- How fast is the car going during the last half hour?
- When is the car traveling fastest?
- When is the car traveling slowest?
- What is the slope of the graph between $t = 90$ and $t = 120$?

Note that the answers to questions b and e above are the same. Slope is the change in y (Δy) divided by the change in x (Δx). In this case, the change in y is the change in distance; the change in x is the change in time. So, the slope of the graph gives the car's rate. Since the graph is a piecewise linear function, finding the rate is easy. It is possible to find the rate of the car at a particular moment since the function is piecewise linear.

- What is the speed of the car at $t = 50$ minutes?
- What is the speed of the car at $t = 72$ minutes?

Example 3: Consider a similar function.



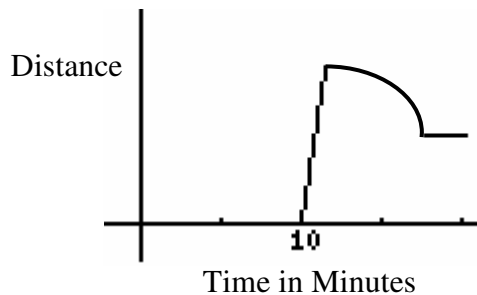
- What is the average rate over the two hours?
- How fast is the car going between $t = 20$ and $t = 30$?
- When is the car traveling fastest?
- When is the car traveling slowest?
- How fast is the car going between $t = 80$ and $t = 120$?
- Why is the segment between $t = 80$ and $t = 120$ negative?
- What is the speed of the car at $t = 10$ minutes?
- What is the speed of the car at $t = 71$ minutes?

Finding the rate of change for a linear or piecewise linear function is easy, because it remains the same over an interval. If the slope of the line is positive, that means the change is also positive, i.e., as x increases, y also increases. If the slope of the line is negative that means the change is negative, i.e., as x increases, y decreases. If there is no change, the slope is zero and the line is horizontal.

Some situations cannot be described by linear functions.

Example 4: Sketch the graph of a function that represents the motion described: Mrs. Eagen went skiing. She waited for the ski lift at the bottom of the hill for 10 minutes, rode up for 3 minutes and then skied downhill. She lost control and began to go faster and faster, until she fell. (a fictitious story because Mrs. Eagen does not fall when she skis.)

Solution: For the first 10 minutes, there was no change in distance; i.e., a horizontal line. For the next 3 minutes, there was a constant rate as she went up on the lift. Then she began slowly and then started speeding up; i.e., a curve that gets steeper. Then the curve abruptly stops.



In the last example, how can Mrs. Eagen's speed be determined at any moment while she is actually skiing? If she was wearing a speedometer, it would be easy to determine the speed. The speedometer, on Mrs. Eagen's skis, on a bike, a car, or a plane, measure the rate of change of the distance with respect to time at any moment. It is often useful to be able to measure this rate. It is difficult to do with a curve.

Example 5: Recall another type of motion problem that we have studied: projectile motion. Suppose an object is projected from the top of a 96 foot platform and that its path is defined by the equation $y = 96 + 80t - 16t^2$

- What is the average speed from $t = 0$ to $t = 2.5$ seconds?
- What is the speed at $t = 2.5$ seconds?
- What is the average speed from $t = 2.5$ to $t = 6$ seconds?

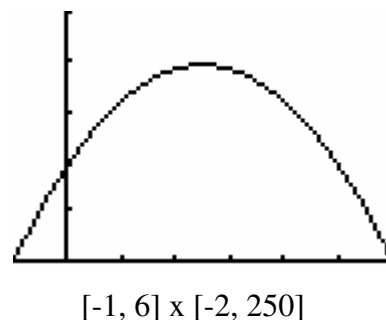
Solution: A table of values for the function and the graph of the function are pictured.

X	Y1
0	96
.5	132
1	160
1.5	180
2	192
2.5	196
3	192

X=0

X	Y1
3	192
3.5	180
4	160
4.5	132
5	96
5.5	52
6	0

X=6

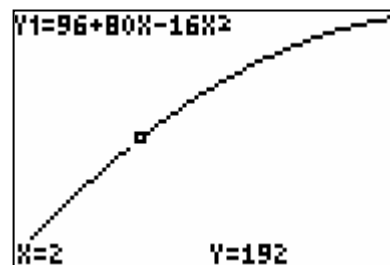


We can again find the average speed by finding $\frac{\Delta y}{\Delta t}$, but in this case it will be difficult to find the speed at a precise moment in time.

Suppose we wish to find the speed at $t = 2$ seconds. By using the formula, we can find the average change in increasingly small increments. Consider the table values with a $\Delta x = 0.1$ and the graph with the same window settings as the table.

X	Y1
1.8	188.16
1.9	190.24
2	192
2.1	193.44
2.2	194.56
2.3	195.36
2.4	195.84

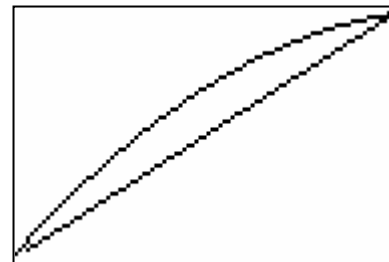
X=1.8



The rate of change in the window is $\frac{195.84 - 188.16}{2.4 - 1.8} = \frac{7.68}{0.6} = 12.8$ ft/sec.

How close is this to the average speed at $t = 2$ seconds? Consider the fact that this rate of change is the slope of the line between the points (1.8, 188.16) and (2.4, 195.84). The line is pictured on the graph at the right.

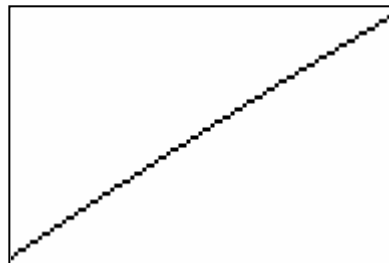
There is quite a difference between the line and the curve.



Suppose the table is reset with $\Delta x = 0.01$ and the graph redrawn and the rate recalculated.

X	Y1
1.97	191.51
1.98	191.67
1.99	191.84
2	192
2.01	192.16
2.02	192.31
2.03	192.47

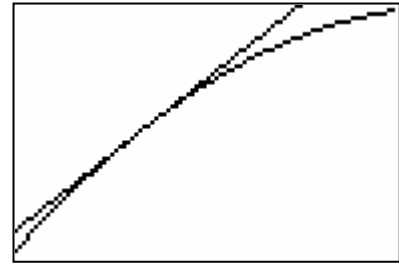
X=1.97



The rate of change in the window is $\frac{192.47 - 191.51}{2.03 - 1.97} = \frac{0.96}{0.06} = 16$ ft/sec.

Is this the rate of change at the moment $t = 2$ seconds? When the line through the points (1.97, 191.51) and (2.03, 192.47) is drawn in this window, it is not possible to distinguish between the line and the curve. The straight line has become an approximation of the curve. The slope of the line indicates the change in rate of the curve.

If the line through the points (1.97, 191.51) and (2.03, 192.47) is drawn in the previous window, it appears as shown at the right.



The line appears to be tangent to the curve. So, in order to find the rate of change at an instantaneous moment on a curve, we can use the tangent line to the curve at the point to approximate that rate of change.

What this example has shown is that if you choose any point on a curve and magnify the graph sufficiently at that point, the result is a straight line (or what appears to be the tangent line to the curve at that point). This phenomenon is called **local linearity**. We can summarize the results as follows:

The rate of change of a function is given by the slope of its graph. If the graph is a curve, the slope at a point P can be approximated by taking another point Q very close to it on the curve and finding the slope of the line joining the two points. By taking these two points very close together, you can make this approximation as close to the true value of the slope as you wish.

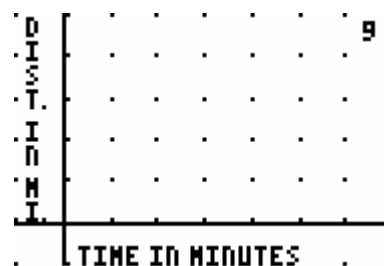
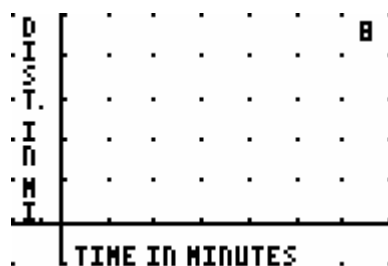
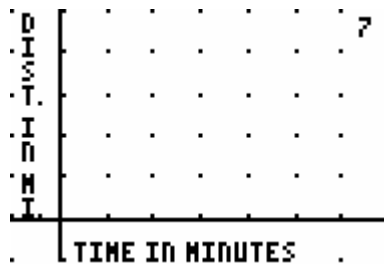
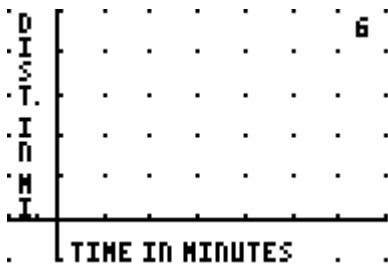
Homework Exercises Part 1: Rates of Change

For each of the following, write an expression for the rate of change for the given situation.

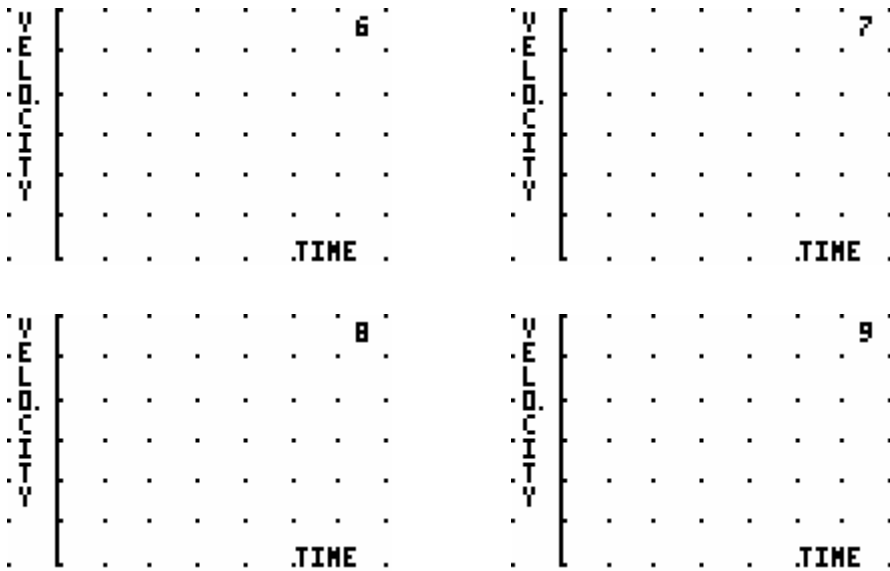
- The area (A) of a square is decreasing as the length of the side (s) decreases from $s = 4$ in. to $s = 1$ in.
- The volume (V) of a sphere decreases as the radius (r) of the sphere decreases from $s = 9$ cm. to $s = 5$ cm.
- The surface area (A) of a cube decreases as the length of the side (s) of the cube decreases from 8 ft. to 3 ft.
- The area of an equilateral triangle (A) decreases as the length of a side (s) decreases from $s = 12$ cm. to $s = 3$ cm.
- The population (P) of a city is increasing with respect to time (t). Express the change in the population from 1990 to 2003. This answer will require a variable expression rather than a numerical expression as in #1-4 since we do not know the function that models the population of the city. See example #1 on page 1.

Draw a **distance-time graph** representing the motion described on the grids provided. Be sure to label axes scale appropriately. A distance-time graph is often called a **position-time graph**.

- Drive 60 mph for 10 minutes; stop for 10 minutes; drive 30 mph for 20 minutes. (Don't forget to change minutes to hours.)
- Drive 30 mph for 10 minutes; drive 60 mph for 10 minutes; turn around and drive 20 mph back to the starting point.
- Drive 30 mph for 5 minutes; stop for 5 minutes; drive 30 mph for 5 minutes; stop for 5 minutes; continue this pattern for one hour.
- Drive 30 mph for 20 minutes; turn around and drive 30 mph back towards your starting point for 10 minutes; turn around and walk away from the starting point again for 10 minutes.



For the same scenarios in #6-9 draw a **velocity-time graph** representing the motion described on the grids provided. Draw the horizontal axis. Be sure to label axes appropriately with units and scale.



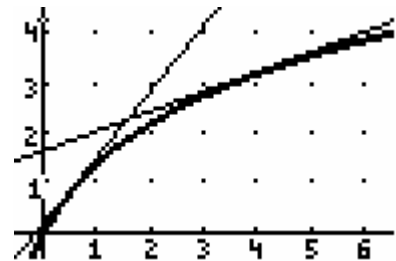
Draw a **distance-time graph** representing the motion described in #10-12. Be sure to label axes appropriately with title and units and scale.

10. John was walking to the school bus stop and saw the bus coming. He ran as fast as he could towards the stop but the bus moved off before he got there. He walked slowly all of the way to school.
11. Mr. Torbert went downhill on his roller blades, speeding up as he went. At the bottom of the hill he ran into the curb and came to a sudden stop. Ouch.
12. Mrs. Montgomery pushes her daughter Jacqueline on a swing until it is swinging very high. Then she stops pushing and the swing gradually slows down and stops.

Part 2: Slope or f' graphs

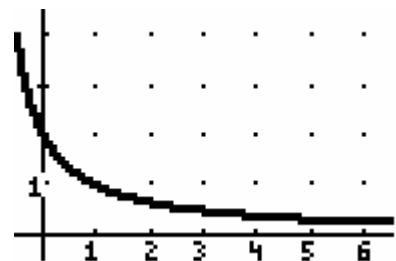
In the previous set of exercises, the velocity-time graphs you sketched for #6-9 are called slope functions or f' functions (read f prime). This graph is important because it represents the rate of change of the original function. In science experiments, we often collect data but do not have an algebraic expression for the data. Though we are often able to find a function that represents the data, it is sometimes impossible. It is still important to analyze the data and find where reactions or changes are occurring most rapidly. This can be done by observing the graph and estimating the slopes of the tangent lines.

Example 1: Look at the graph of $y = f(x)$ at the right and analyze the rate of change in the graph as you move from left to right.

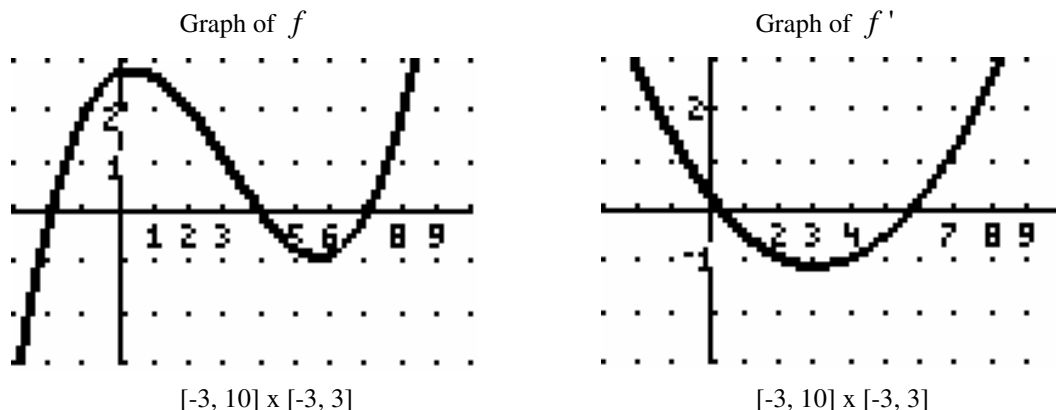


Solution: The curve is initially very steep and the slope of the tangent line is positive. At $x = 1/2$, the slope is approximately 1.5. At $x = 4$, the slope has decreased and is approximately 0.5.

The slope graph or f' graph would like the one pictured below.



Example 2: Study the graph of f and its f' graph below.



The f' graph tells how the slopes of the tangent lines to the f graph behave. At $x=2$, for example, the slope appears to be approximately -1 . For this reason, the f' graph has a value of -1 at $x=2$. There are several relationships between the two graphs to note, and some new definitions to learn.

The graph of f rises to the left of $x=0$, falls between $x=0$ and $x \approx 5.5$, and rises again to the right of $x \approx 5.5$. The sign of $f'(x)$ indicates whether the line tangent to the f graph at x points up or down. On the f' graph, the sign changes at $x=0$ and $x \approx 5.5$. It is positive (above the x -axis) to the left of $x=0$, negative (below the x -axis) between $x=0$ and $x \approx 5.5$, and is again positive to the right of $x \approx 5.5$. So, at $x=0$ and $x \approx 5.5$, f itself changes direction. When a graph is rising, we say that the function **increases**; when the graph is falling, we say the graph **decreases**.

At $x=0$ and $x \approx 5.5$, the graph of f has horizontal tangent lines. Such points are called **stationary points** of f . The value of $f'(x)$ at these points is 0 since the slope of a horizontal line is 0 . These are the x -intercepts of the $f'(x)$ graph.

Why does the graph of $f(x)$ reach a minimum value at $x=3$? Study the graph of f carefully. At $x=3$ on the graph of f , the graph is at its steepest. Thus, this is the smallest value that the slope will reach.

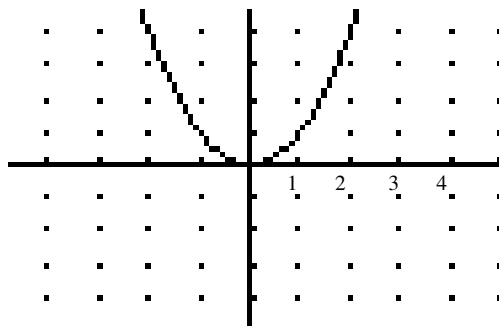
Follow these steps when you sketch the $f'(x)$ graph.

1. Describe how the slope changes from left to right along the curve. Note the intervals where the slope is positive, where the slope is negative, and where the slope is zero.
2. Write down the changes as they occur from left to right, noting where the x -intercepts occur, where the graph is positive (above the x -axis) and where it is negative (below the x -axis).
3. Check your $f'(x)$ graph with the original graph.

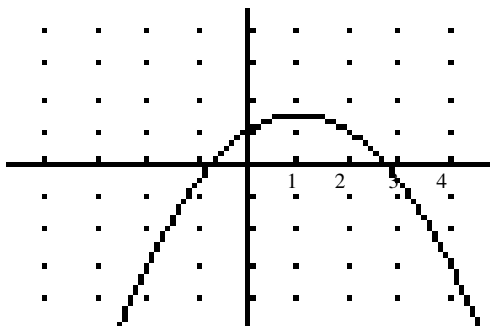
Homework Part 2: Slope or f' graphs

For problems 1 - 2, estimate the slope of the curve at the point with the indicated x-coordinate.
All viewing windows are $[-5, 5] \times [-5, 5]$.

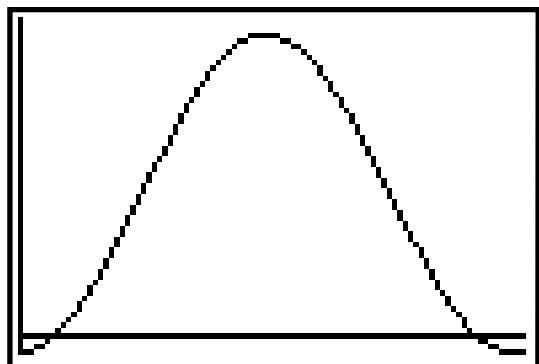
1. a) $x = 1$ b) $x = -1$



2. a) $x = 0$ b) $x = 1$ c) $x = 2$



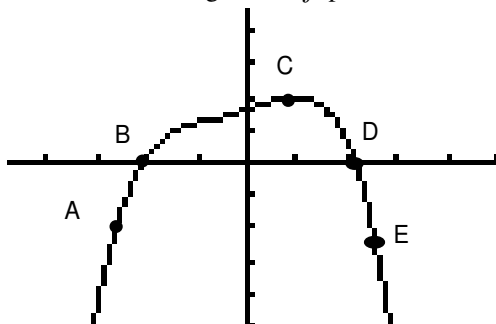
3. The window below shows the Fahrenheit temperature in Washington, D. C. for a typical 365 day period from January 1 to December 31. Answer the questions by estimating the slopes on the graph in degrees per day.



$[0, 365] \times [-5, 105]$

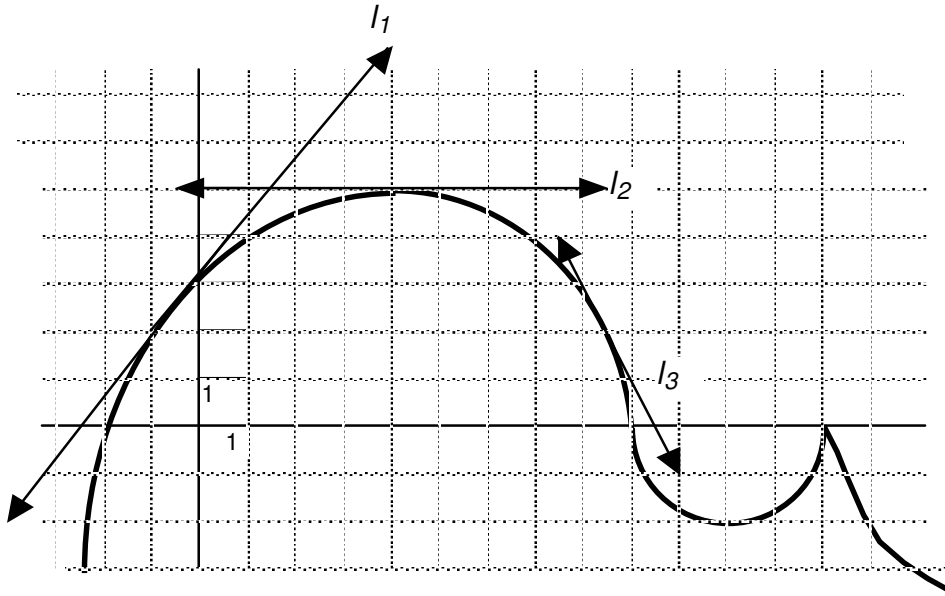
- On about what date is the temperature increasing at the fastest rate? What is that rate?
- Do there appear to be days on which the temperature's rate of change is zero? If so, which ones?
- During what period is the temperature's rate of change positive? negative?

4. Refer to the diagram of f pictured below.



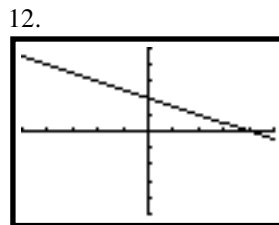
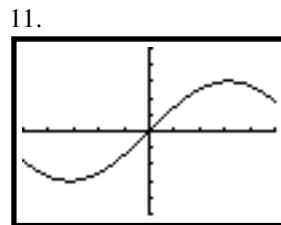
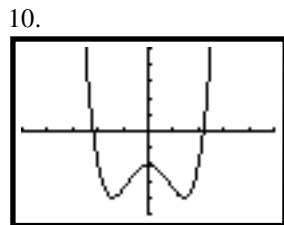
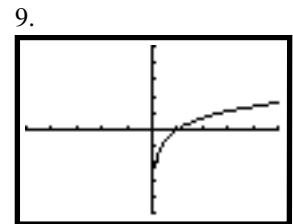
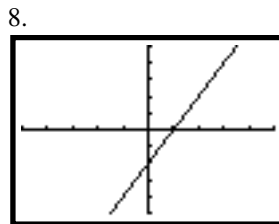
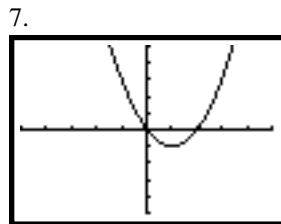
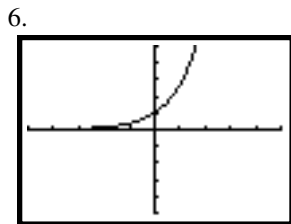
- Which point(s) are stationary points?
- At what points does f have a horizontal tangent line?
- At what labeled points is the slope of the curve positive?
- At what labeled points is the slope of the curve negative?
- Which labeled point has the greatest slope, i.e., most positive slope?
- Which labeled point has the least, i.e., most negative slope?
- Where is f increasing?
- Where is f decreasing?
- Sketch the graph of the $f'(x)$ graph.

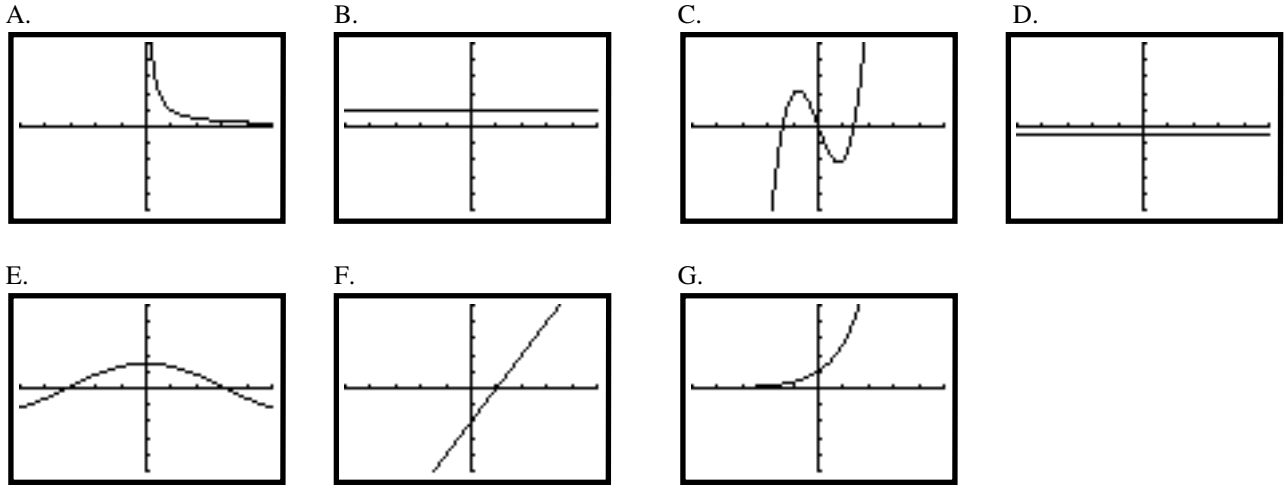
5. The diagram below shows the graph of a function, $f(x)$. The three labeled lines are tangent to $f(x)$.



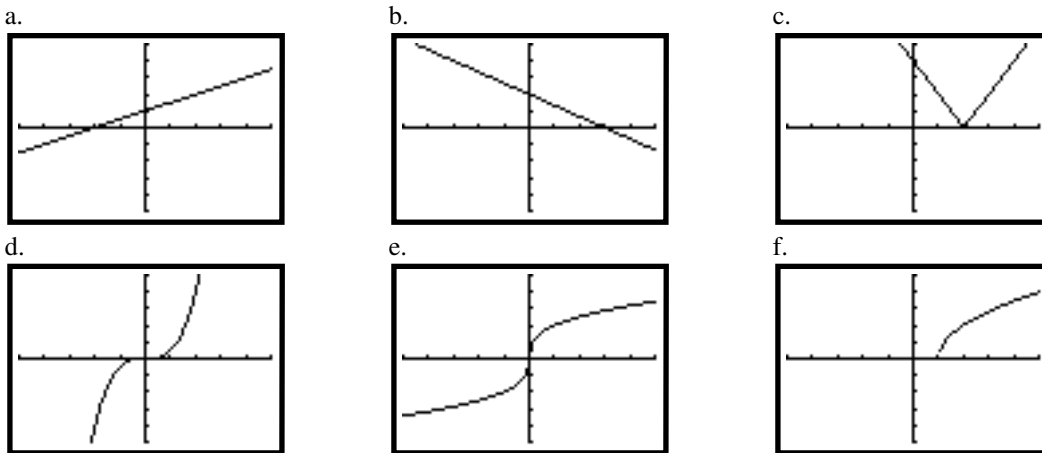
- What is $f'(-1)$?
- What is $f'(4)$?
- What is $f'(8.5)$?
- For what values of x is $f'(x) = 0$?
- For what values of x is $f'(x) < 0$?
- For what values of x is $f'(x) > 0$?
- What does the slope of $f(x)$ appear to be at $x = 13$?
- True or False: If a line intersects a curve in just one point, then the line is tangent to the curve.
- True or False: If a line is tangent to a curve, then the line intersects the curve in just one point.
- Graph $f'(x)$.

For each $f(x)$ in 6 - 12, match the graph with its $f'(x)$ graph A - G. All windows are $[-5, 5] \times [-5, 5]$.





13. Sketch the $f'(x)$ graph for each of the following graphs. All windows are $[-5, 5] \times [-5, 5]$.



14. Sketch the graph of a function f with the following characteristics:

x	$f(x)$	$f'(x)$
-3	1	-3
-1	-1	0
1	-1	-1
3	2	-2

15. Sketch a function whose slope is everywhere positive and it is increasing gradually.
16. Sketch a function whose slope is everywhere positive and it is increasing sharply.
17. Sketch a function whose slope is everywhere negative and it is decreasing gradually.
18. Sketch a function whose slope is everywhere negative and is decreasing sharply.

19. Use the chart of values for a function f to answer the questions below.

x	-3	-2	-1	0	1	2	3	4	5	6	7
$f(x)$	15	12	9	8	8	10	14	20	22	30	36

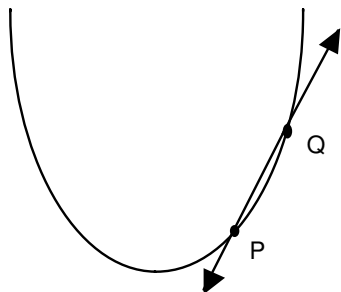
- a. Where is the rate of change of $f(x)$ positive?
- b. Where is the rate of change of $f(x)$ negative?
- c. Where does the rate of change of $f(x)$ appear to be greatest?

Part 3: The Tangent Line

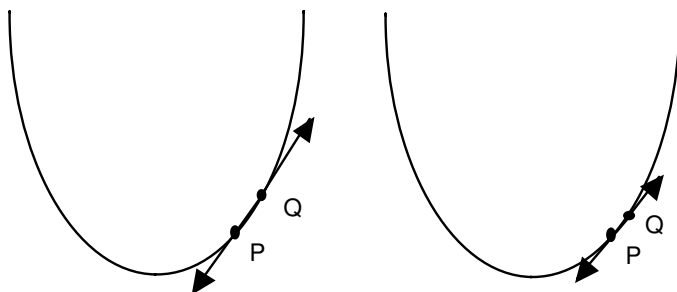
Recall the projectile motion from Part 1 and the conclusion that was reached:

The rate of change of a function is given by the slope of its graph. If the graph is a curve, the slope at a point P can be approximated by taking another point Q very close to it on the curve and finding the slope of the line joining the two points. By taking these two points close enough together, you can make this approximation as close to the true value of the slope as you wish.

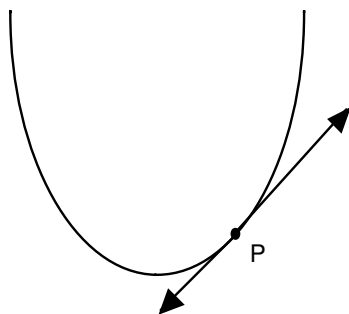
When we talk about joining two points on the curve, we are drawing a secant line to the curve.



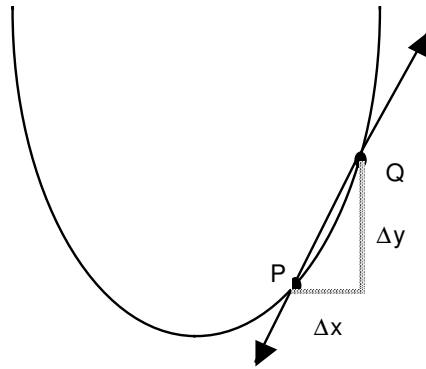
We move Q closer to P



until the tangent line is reached at the point P. (*Tangere* is Latin for “to touch”.)



What is happening is that Δx is being made smaller as Q approaches P



The slope of the secant line with $P(x, f(x))$ and $Q(x + \Delta x, f(x + \Delta x))$ is

$$\text{slope of secant line} = \frac{f(x + \Delta x) - f(x)}{x + \Delta x - x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

As Q approaches P, Δx is approaching 0. This is a limit problem and the slope of the tangent line, the line which defines the rate of change at P is the limit of the slope of the secant line:

$$\text{slope of tangent line} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Example 1: Find the slope of the tangent line to the curve $y = 3x^2$ at the point (1, 3) using the limit definition of the slope of the tangent line to a curve. Then write the equation of this tangent line.

Solution: In this problem $x = 1$. Substitute into the formula above.

$$\begin{aligned} \text{slope of the tangent line} &= \lim_{\Delta x \rightarrow 0} \frac{3(1 + \Delta x)^2 - 3(1)^2}{\Delta x} = \\ & \lim_{\Delta x \rightarrow 0} \frac{3(1 + 2\Delta x + \Delta x^2) - 3}{\Delta x} = \\ & \lim_{\Delta x \rightarrow 0} \frac{3 + 6\Delta x + 3\Delta x^2 - 3}{\Delta x} = \\ & \lim_{\Delta x \rightarrow 0} \frac{6\Delta x + 3\Delta x^2}{\Delta x} = \\ & \lim_{\Delta x \rightarrow 0} (6 + 3\Delta x) = \\ & 6 \end{aligned}$$

Therefore, the slope of the tangent line to the curve $y = 3x^2$ at the point (1, 3) is 6. Using the point slope formula, the equation of the tangent line to the curve is $y - 3 = 6(x - 1)$.

Example 2: Find a general formula for the slope of the tangent line to any point on the curve $y = 1 - 2x + x^2$. Use the general formula to find the slope of the tangent line at $x = 0$, $x = -4.5$, and $x = 19.8$.

$$\begin{aligned} \text{Solution: slope of the tangent line} &= \lim_{\Delta x \rightarrow 0} \frac{[1 - 2(x + \Delta x) + (x + \Delta x)^2] - [1 - 2x + x^2]}{\Delta x} = \\ &= \lim_{\Delta x \rightarrow 0} \frac{[1 - 2x - 2\Delta x + x^2 + 2x\Delta x + \Delta x^2] - 1 + 2x - x^2}{\Delta x} = \\ &= \lim_{\Delta x \rightarrow 0} \frac{-2\Delta x + \Delta x^2 + 2x\Delta x}{\Delta x} = \\ &= \lim_{\Delta x \rightarrow 0} (-2 + \Delta x + 2x) = \\ &= -2 + 2x \end{aligned}$$

Therefore, the formula for the slope of the tangent line to any point on the curve is $m = -2 + 2x$. At $x = 0$, the slope is -2 . At $x = -4.5$, the slope is -11 . At $x = 19.8$, the slope is 37.6 .

Homework Part 3: The Tangent Line

- Consider the function $y = 1 - x^2$.
 - Sketch the graph carefully. On your graph draw the tangent line at the point $(2, -3)$. Estimate the slope of this tangent line.
 - Calculate the slope of the secant line through $(2, -3)$ and $(2.01, (1 - 2.01^2))$.
 - Use the limit definition of the slope of a tangent line to find the exact slope of the tangent at the point $(2, -3)$.
- Consider the function $y = 2x^3$.
 - Sketch the graph carefully. On your graph draw the tangent line at the point $(1, 2)$. Estimate the slope of this tangent line.
 - Calculate the slope of the secant line through $(1, 2)$ and $(1.01, 2(1.01^3))$.
 - Use the limit definition of the slope of a tangent line to find the exact slope of the tangent at the point $(1, 2)$.
- Use the limit definition of the slope of a tangent line to find the general formula for the slope of any tangent line to the curve $y = 3x^2 - x$. Use this formula to find the slopes at $x = 0$, $x = \frac{1}{2}$ and $x = -10$.
- Use the limit definition of the slope of a tangent line to find the general formula for the slope of any tangent line to the curve $y = \frac{1}{x}$. Use this formula to find the slopes at $x = -1$, $x = \frac{1}{2}$ and $x = -10$.
- Using the limit definition of the slope of a tangent line, find the equation of the tangent line to the curve $y = x^3 + x$ at the point $(1, 2)$.
- Using the limit definition of the slope of a tangent line, find the equation of the tangent line to the curve $y = \sqrt{x}$ at the point $(4, 2)$.
- Use the limit definition of the slope of a tangent line to find the general formula for the slope of any tangent line to the curve $y = x^2$. For what values of x , if any is $f'(x)$ positive? zero? negative?

8. Use the limit definition of the slope of a tangent line to find the general formula for the slope of any tangent line to the curve $y = x^3$. For what values of x , if any, is $f'(x)$ positive? zero? negative?
9. Use the limit definition of the slope of a tangent line to find the slope of the curve $y = \sqrt[3]{x}$ at $x = 0$. Describe the tangent line at $x = 0$. Graph the function on your calculator. Does your answer make sense? Study the curve immediately to the right and to the left of 0 and note how steep the curve is.
10. Sketch the graphs of $y = 2x^2$ and $y = 2x^2 + 5$ on the same set of axes. What appears to be true about the slopes of the tangent lines to the two graphs at the point $x = 0$? $x = 2$? $x = -1$? Use the limit definition of the slope of a tangent line to find the general formula for the slope of any tangent line to each of the two curves in order to show that adding a constant value, c , to any function does not change the value of the slope of its graph at any point.

Part 4: The Definition of Derivative

The limit definition of the slope of a tangent line can be generalized in order to include any rate of change including velocity, growth rate, and other quantities mentioned earlier. The name that is used to indicate a rate of change function is derivative.

The derivative of a function f is another name for the function f' whose value is given by

$$\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

provided the limit exists.

If the limit does exist, we say that f is **differentiable**. Finding a derivative is called **differentiation**.

Finding the slope of a tangent line to a curve is only one of many applications of differentiation. Another common application is finding velocity as we did earlier. If you drive a car 80 miles over a 2 hour period your average velocity is 40 mph. But what is the velocity at any instant in time, i.e., your **instantaneous velocity**. Recall the projectile motion problem from Part 1. We found the velocity at $t = 2$ seconds by zooming in on the graph and finding the slope of the curve at that point. We found the velocity at that instant to be 16 ft/sec. Confirm this finding by using the definition of derivative.

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \frac{[96 + 80(2 + \Delta t) - 16(2 + \Delta t)^2] - [96 + 80(2) - 16(4)]}{\Delta t} &= \\ \lim_{\Delta t \rightarrow 0} \frac{96 + 160 + 80\Delta t - 16(4 + 4\Delta t + \Delta t^2) - 192}{\Delta t} &= \\ \lim_{\Delta t \rightarrow 0} \frac{96 + 160 + 80\Delta t - 64 - 64\Delta t - 16\Delta t^2 - 192}{\Delta t} &= \\ \lim_{\Delta t \rightarrow 0} \frac{16\Delta t - 16\Delta t^2}{\Delta t} &= \\ \lim_{\Delta t \rightarrow 0} (16 - 16\Delta t) &= 16 \end{aligned}$$

So, we have confirmed analytically that the velocity at $t = 2$ seconds is 16 ft/sec.

Another common application of derivatives is the study of population growth or decline.

Example 1: A laboratory is testing a new pesticide. The scientists spray a sample of gypsy moths with the new pesticide and find that the population continued to grow for a while but then stopped growing and began to decline. The size of the population at time t (days) was $p(t) = 100,000 + 100t - 1000t^2$. Find the growth rates at $t = 0$, $t = 5$ and $t = 10$ days.

Solution: Since you are asked to find the growth rate at more than one t , it would be most useful to find the general formula for the derivative and then substitute each of the values.

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \frac{[100,000 + 100(t + \Delta t) - 1000(t + \Delta t)^2] - [100,000 + 100t - 1000t^2]}{\Delta t} &= \\ \lim_{\Delta t \rightarrow 0} \frac{100\Delta t - 1000(t^2 + 2t\Delta t + \Delta t^2) + 1000t^2}{\Delta t} &= \\ \lim_{\Delta t \rightarrow 0} \frac{100\Delta t - 1000t^2 - 2000t\Delta t - 1000\Delta t^2 + 1000t^2}{\Delta t} &= \\ \lim_{\Delta t \rightarrow 0} \frac{100\Delta t - 2000t\Delta t - 1000\Delta t^2}{\Delta t} &= \\ \lim_{\Delta t \rightarrow 0} (100 - 2000t - 1000\Delta t) &= 100 - 2000t \end{aligned}$$

Therefore, for $t = 0$ the rate of change is 100 gypsy moths; for $t = 5$, the rate of change is -9900 gypsy moths; and, for $t = 10$ the rate of change is -19900 gypsy moths/day.

Note in the velocity and population problems, there are units associated with the answers. It is very important to indicate and understand the meaning of the units in a derivative problem. If you are told that the cost of building a house is determined by the number of square feet in the house, you might represent the function as $C = f(A)$ where C is the cost and A is the area. What would $f'(1000) = 20000$ mean? There is a change in cost with respect to the change in area. This expression indicates that for a 1000 square foot increase in the size of the house, there is a cost increase of \$20000.

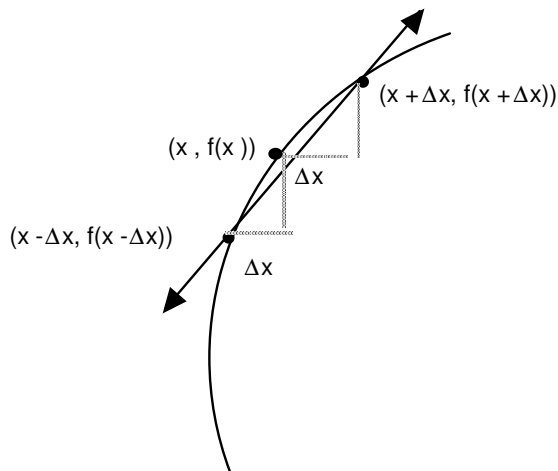
Homework Part 4: The Definition of Derivative

- An object travels along a line so that its position s is $s = 3t^2 + 5$ meters after t seconds.
 - What is the average velocity on the interval $3 \leq t \leq 4$?
 - What is the average velocity on the interval $3 \leq t \leq 3.01$?
 - Find the instantaneous velocity of the object using the definition of derivative at $t = 3$ seconds.
- A bacteria culture is growing so that it has a mass $m = 0.25t^2 + 2$ after t hours.
 - How much did the culture grow during the interval $5 \leq t \leq 6$ hours?
 - How much did the culture grow during the interval $5 \leq t \leq 5.01$ hours?
 - Find the instantaneous growth rate using the definition of derivative at $t = 5$ hours.
- Suppose that a manufacturer's profit from producing a new toy is $P(x) = 0.5x - 0.001x^2$ where x is the number of toys produced. As more toys are produced there is a change in the profit. Using the definition of derivative find the instantaneous change in profit when $x = 10$ and $x = 100$.

4. A school is hit by a measles outbreak. The school nurse estimates that t days after the beginning of the outbreak, the number of persons sick with the measles is given by $m(t) = 60t^2 - t^3$ provided that $0 \leq t \leq 21$. At what rate is the flu spreading at time $t = 10$? Use the definition of derivative to solve this problem.
5. The rate of change of electric charge with respect to time is called current. Suppose $0.4t^3 + t$ coulombs of charge flow through a wire in t seconds. Find the current in amperes (coulombs per second) after 3 seconds. Use the definition of derivative to solve this problem.
6. The temperature, T , in degrees Fahrenheit, of a cold cup of water placed in a microwave oven is given by $T = f(t)$ where t is the time in seconds since the cup was placed in the microwave and the microwave started.
- What is the sign of $f'(t)$?
 - What are the units of $f'(20)$? What is the practical meaning of $f'(20) = 40$?
7. Domino's Pizza's revenue from sales, R (measured in thousands of dollars), is a function of its advertising budget, A , also measured in thousands of dollars. Let $R = F(A)$.
- What does the company hope is true about the sign of $f'(A)$?
 - What does the statement $f'(100) = 20$ mean in practical terms? What about $f'(100) = \frac{1}{2}$?

Part 5: The Numeric Derivative

The definition of derivative given in the previous section is not the only definition of derivative. In the given definition, we approached the point where we wish to find the derivative from one side and made the distance Δx approach zero. Another common definition is called the symmetric difference quotient. This definition approaches the point where we wish to find the derivative from both sides.

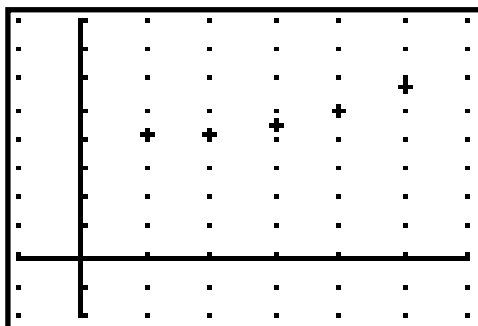


The limit definition of the symmetric difference quotient is
$$\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}$$

Example 1: Evaluate $f'(4)$ from the following table which shows the values of the function f .

x	1	2	3	4	5	6
f(x)	4.2	4.1	4.2	4.5	5	5.7

The points are plotted below. Using the grid, estimate the slope at $x = 4$.

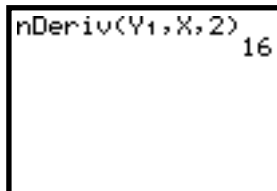
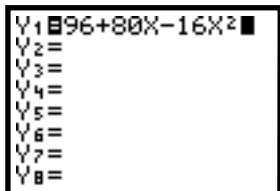


$[-1, 6] \times [-2, 8]$

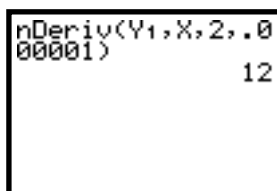
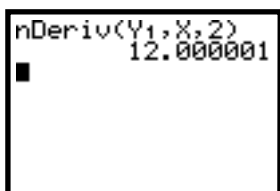
Using the definition from the previous section and the chart of values, if the derivative is found as you approach x from the right, you get $f'(4) \approx \frac{f(5) - f(4)}{5 - 4} \approx 0.5$. If the derivative is found as you approach x from the left, you get $f'(4) \approx \frac{f(4) - f(3)}{4 - 3} \approx 0.3$. In these two cases, Δx is quite large ($\Delta x = 1$). Would the average of these two values give a better approximation of the derivative at $x = 4$? The average of the two results is $f'(4) \approx \frac{0.5 + 0.3}{2} \approx 0.4$. This is the value that the symmetric difference would yield.

The symmetric difference quotient is the one programmed into the TI-83. The operation is called nDeriv under the MATH menu. There are four parameters required for nDeriv: a function, generally stored in Y1; the variable the derivative is found with respect to, generally x ; the value at which you wish to find the derivative; and, Δx (if you do not enter Δx , the default value is used: 0.001).

Once again, recall the projectile motion problem from Parts 2 and 4. The object traveled in a path defined by the equation $y = 96 + 80t - 16t^2$. We found the instantaneous velocity of the object at $t = 2$. In order to use the calculator for this problem, enter this equation in Y1 and enter the nDeriv command in the home screen. The default for Δx is used below.



nDeriv gives an approximation of the derivative. In this case above, the exact answer was found. Try $Y1 = x^3$. The first window yields the nDeriv approximation. If Δx is decreased to 0.000001, the result in the second window is exact.



So, it is important to consider the result the calculator yields and determine its accuracy.

It is also possible for your calculator to graph the derivative of a function. That is how the pictures were achieved in this packet. The graphs on page 6 were achieved with the following commands.

```

Y1=05(X+2)(X-4)
(X-7)
Y2=fnDeriv(Y1,X,X)
)
Y3=
Y4=
Y5=
Y6=
    
```

Finally, it is possible to find the derivative at a particular point on a curve after the function is graphed using the 2nd CALC menu and choosing 6: dy/dx.

```

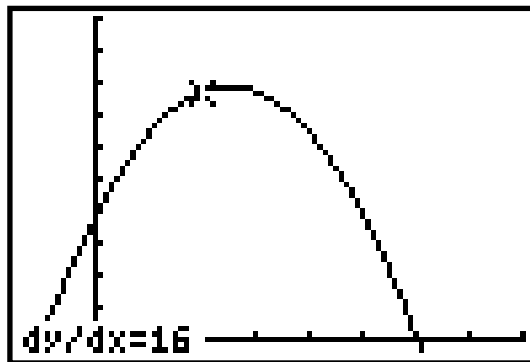
CALCULATE
1:value
2:root
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx
    
```

Once again, consider the projectile motion problem. Graph $y = 96 + 80t - 16t^2$ in the given window:

```

WINDOW FORMAT
Xmin=-1.4
Xmax=8
Xscl=1
Ymin=-10
Ymax=250
Yscl=25
    
```

Now, choose 2nd CALC 6: dy/dx. Move the cursor to $x = 2$ and press enter:



There are some problems with using the calculator to determine the numeric derivative of a function. We will examine these difficulties when we study the rules for differentiability in the following chapter.

Homework Part 5: The Numeric Derivative

For problems 1 - 4, find the numeric derivative (using NDer), accurate to three decimal places, and use it to write an equation for the tangent line to the curve at the given point. Graph each of the curves and the tangent line to the curve at the indicated point in order to answer question 5.

1. $y = 2x^3 - 5x - 2$ at $x = 1.5$

3. $y = \frac{x^2 - 4}{x^2 + 2}$ at $x = 2$

2. $y = \sqrt{4 - x^2}$ at $x = -1$

4. $y = x \sin x$ at $x = 2$

5. Considering what you found in problems 1 - 4, can a tangent line to a curve cross the curve more than once?

6. Suppose that the profits from a business can be determined by the function $y = 2350 \sin^2(0.00001x^2)$. If the business is producing 20,000 units per week, is the profit increasing or decreasing and at what rate?

7. Suppose the velocity of an object at time t is given by $v = \frac{20 \cos^2(2x^3 + x^2 + 1)}{6x^2 + 5}$. Find the change in velocity with respect to time (the acceleration) at $t = 2.5$ seconds.

8. The weight in ounces of a fetus at time t is $W(t) = 0.1t^2 - 0.08t$, where t is measured in weeks. Find the rate of growth of the fetus at $t = 35$ weeks. Confirm your answer analytically (i.e., with the definition of derivative).

9. A deer population is modeled by the equation $P(t) = 3000 + 600 \sin\left(\frac{\pi}{6}t - 2\pi\right)$ where t is measured in months.

- Sketch the population for one year. Tell what window you used.
- When during the year is the population a maximum? When is the population at a minimum?
- When is the population growing fastest? When is it decreasing fastest?
- Estimate how fast the population is changing on June 1.

Homework Review Exercises

For problems 1 - 5, draw a function graph and a rate graph to illustrate each of the following situations. Label axes.

- Water is evaporating from a tank at a constant rate.
- A child's temperature is rising, but penicillin is injected and takes effect.
- Water is being poured into a tank at 3 gallons per minute, but the tank is also leaking at a rate of $\frac{1}{2}$ gallon per minute.
- Inflation held steady during the last 12 months, but is expected to rise more and more rapidly in the next several years.
- The angle the Leaning Tower of Pisa makes with the vertical is increasing more rapidly now than in the past.
- Suppose there is another tanker accident and an oil spill occurs.
 - Some days after the spill, it was reported that an oil slick covered an area 30 miles long and 10 miles wide. One day later it was estimated that the slick has increased to 36 miles long and 12 miles wide. What was the average rate of change of the area of the slick with respect to time during this one day? What assumptions did you make in order to answer this question?
 - Assume the spread of the oil slick is a linear model. Draw a graph showing the area covered by the oil as a function of time and find an algebraic formula for this function.
 - Estimate how long the oil had been spreading before the first measurement occurred.
 - Explain the relationship between your graph, the formula and the rate at which the slick is spreading.

Tell whether each of the following statements is true or false.

7. The slope of the tangent line to the curve $y = x^2$ is different at every point of the curve.
8. The slope of the tangent line to the curve $y = x^3$ is different at every point of the curve.
9. A tangent line to a curve cannot cross the curve.
10. If the tangent line to the graph of a function $y = f(x)$ is horizontal at $x = c$, then $f'(c) = 0$.
11. The expression $\lim_{\Delta x \rightarrow 0} \frac{(1 + \Delta x)^2 - 1^2}{\Delta x}$ is the expression for the derivative of $f(x) = x^2$ at $x = 1$.

Use the limit definition of derivative to find the general derivative for each of the following:

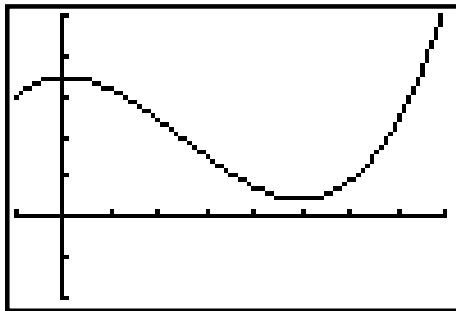
12. $f(x) = x^2 - 2x$ 13. $f(x) = \frac{1}{x-1}$ 14. $f(x) = \sqrt{4-x}$

15. Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent line is parallel to the x-axis. Support your answer analytically.

16. Using the definition of derivative, find an equation of the tangent line to the graph of $y = x^2 + 4x + 1$ at $x = 2$. Confirm your answer by using your calculator.

17. Suppose the temperature T of food in a freezer is given by $T = \frac{600}{t^2 + 2t + 8}$ where t is in hours. Find the rate of change of T with respect to t at $t = 1$, $t = 3$, $t = 5$ and $t = 10$.

18. The graph below pictures the position of an object. Use the diagram to approximate each of the quantities.



$[-1, 8] \times [-20, 50]$

- a. $f'(2)$
- b. $f'(5)$
- c. $f'(7)$
- d. average velocity on $[5, 8]$
- e. Sketch the velocity (derivative) function of the given function.

19. Each of the following pictures shows a function and its derivative. Identify the graphs as f and f' .

