DERIVATIVES: APPLICATIONS

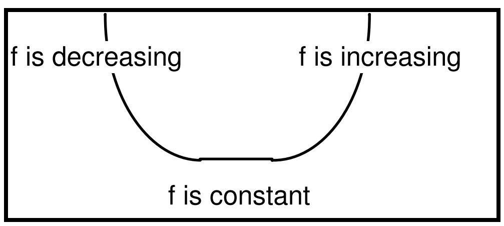
Two useful applications of derivatives have already been discussed: tangent lines and velocity of a position function. There are many applications of derivatives: curve sketching, optimization, mathematical models in economics, biology, medicine and the social sciences. Some of these will be explored in this section.

Part 1: Curve Sketching and Critical Points

When you studied functions and curve sketching earlier in this course, you often found that it was difficult to get a complete graph in a convenient viewing window. There are several important features of a graph to identify: where it is increasing, where it is decreasing, where it reaches a maximum value and where it reaches a minimum value, where it is steepest, etc.

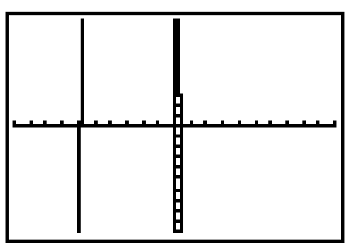
A function is said to be increasing on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

A function is said to be decreasing on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.



Previously, we found the intervals where a function was increasing, decreasing or constant by trial and error: zooming and changing window settings until it was possible to identify the features of a graph. We have found that the derivative can tell us where a function has horizontal tangent lines, where it has vertical tangent lines; it can also tell us where a function is increasing and decreasing and where possible maximum and minimum values occur.

Example 1: Consider the function $f(x) = x^3 - 9x^2 - 81x + 32$. In a standard viewing window, the graph of the function is not helpful in determining the important features of the function:



The first derivative can be used to find where the function is increasing, and where it is decreasing: $f'(x) = 3x^2 - 18x - 81 = 3(x^2 - 6x - 27) = 3(x-9)(x+3)$

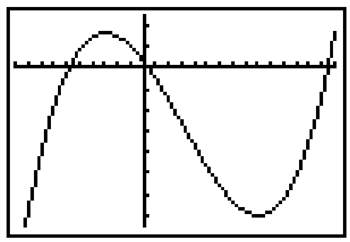
To find where f is increasing, find where f' is positive. To find where f is decreasing, find where f' is negative. The easiest way to do this is to set up a number line. Find where $f'(x) = 0$ by setting the derivative equal to zero and solving the equation: $f'(x) = 3(x-9)(x+3) = 0$ when $x = 9$ and $x = -3$.

Since f' is a polynomial function, it is continuous so f' cannot change sign between these points. To the left of -3, the sign of f' is (-)(-) = +. Between -3 and 9, the sign of f' is (-)(+) = - . To the right of 9, the sign of f' is (+)(+) = +.

This can be summarized on a number line.

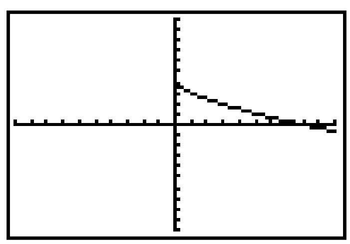
f' __+__|______-_______|__+__
-3   9

So, f is increasing to the left of -3, it is decreasing between -3 and 9, and it is increasing to the right of 9. Evaluating the function at -3 and 9, we find $f(-3) = 167$ and $f(9) = -697$. It is not surprising that a complete graph was not visible in a standard window. Resetting the window to [-10, 15] x [-750, 250] allows viewing of the complete graph.



In the previous example, x = -3 and x = 9 are stationary numbers of the function and the maximum and minimum values of the functions occur at these numbers. In this case, they occurred where $f'(x) = 0$. Maximum and minimum values can occur elsewhere.

<u>Example 2</u>: Consider the function $f(x) = 4 - x^{\frac{2}{3}}$. If this function is entered into Y1 in the following way the graph in the standard viewing window is as shown. Y1 = 4 - x^(2/3).



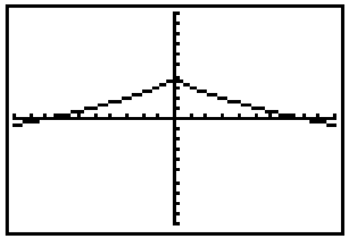
Study the function and derivative to see if this graph makes sense $f'(x) = -\frac{2}{3}x^{-\frac{1}{3}} = \frac{-2}{3\sqrt[3]{x}}$

Both the function and derivative involve taking the cube root. It is possible to take the cube root of a negative or positive number. As a matter of fact, when x is negative, f' is positive, and when x is positive f' is negative.

f' _____+_____|_____-_____
0

That would indicate that the graph should increase to the left of 0 and decrease to the right of 0. It appears that only half of the graph is pictured. So, the pictured graph does not make sense.

Reenter the function as follows: Y1 = 4 - (x^(1/3))^2



The graph and the analysis of the first derivative now coincide.

In this case, the critical point, the value where the graph changed from increasing to decreasing was not where $f'(x) = 0$ but where $f'(x)$ is undefined.

For any function f, a point (c, f(c)) in the domain of f where f ' (c) = 0 or where f ' (c) is undefined is called a critical point of the function. Furthermore, if f ' (c) = 0 the point (c, f(c)) is called a stationary point; if f ' (c) is undefined, the point (c, f(c)) is called a singular point.

Critical points can also indicate where extreme values of functions may occur. Note in Example 1, that the point (-3, 167) is where a local maximum occurs. That is, it is the highest point in an interior interval of the function. Likewise, (9, -697) is where a local minimum occurs, the lowest point in an interior interval of the function. In Example 2, the point (0, 4) is where the local maximum occurs. Not only is it a local maximum, it is the absolute maximum of the function, since 4 is the greatest value the function reaches.

A function f has a relative maximum at an interior point c of its domain if $f(x) \leq f(c)$ for all x in some interval containing c. The relative maximum is the value $f(c)$.

If a relative maximum is the largest value of the function over its entire domain, it is called the absolute maximum.

A function f has a relative minimum at an interior point c of its domain if $f(x) \geq f(c)$ for all x in some interval containing c. The relative minimum is the value $f(c)$.

If a relative minimum is the smallest value of the function over its entire domain, it is called the absolute minimum.

The set of relative maxima and minima are called relative extrema.

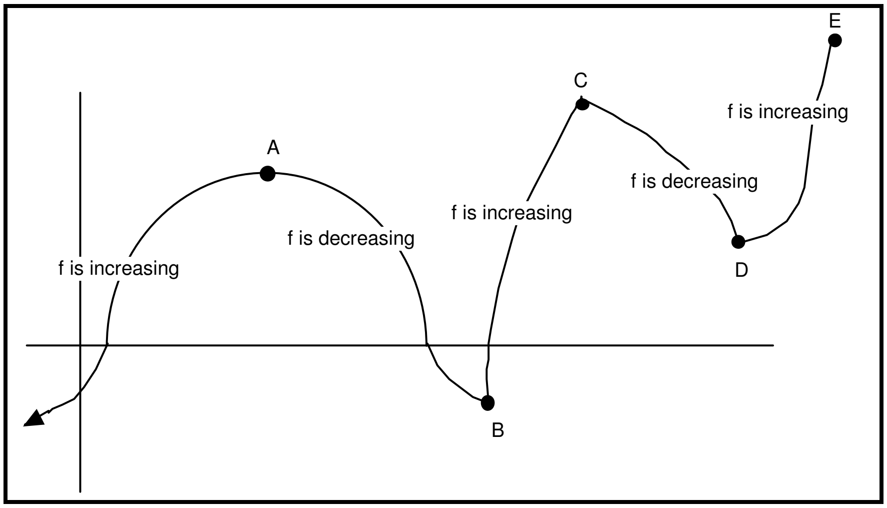
The set of all relative and absolute maxima and minima are called extrema or extreme values.

Finally, an absolute maximum or minimum may occur at the endpoint of the domain of a function. The endpoints of the domain are the final type of critical point of a function.

In summary, the two types of critical points are

1. **stationary points (f' (x) = 0)**
2. **singular points (f ' (x) is undefined)**

And, absolute extrema occur at critical points or endpoints while relative extrema occur may only occur at critical points.



In the diagram, A and C are relative maxima. B and D relative minima. E is the absolute maximum of the function. There is no absolute minimum.

Absolute extreme values occur at critical points or endpoints of a function; relative extrema only occur at critical but not all critical points are extreme value points.

To find relative extrema:

(1) Find $f'(x)$ Set if equal to 0 or find where it does not exist.

(2) Use a $f'(x)$ line to test if the value is a maximum, minimum or neither.

To find absolute extrema: Examine the relative extrema and endpoints and determine the largest or smallest value.

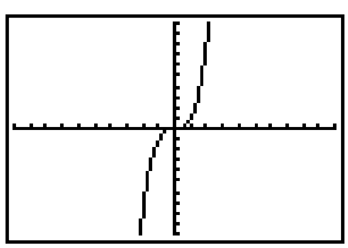
Example 3: Graph and find the critical points of the function $f(x) = x^3$ where the domain is all real numbers. Find where the function is increasing and where it is decreasing.

Solution: Since critical values occur where $f'(x) = 0$ or is undefined, first find $f'(x)$. $f'(x) = 3x^2$. $f'(x)$ is always defined. $f'(x) = 0$ at x = 0.

So, x = 0 is the only point where a minimum or maximum can occur. Consider the $f'(x)$ number line.

f' _____+_____|_____+_____
0

$f'(x)$ is always positive, so it is always increasing. There is no extreme value at x = 0. This is confirmed with the graph.



Homework Part 1: Curve Sketching and Critical Points

Exercises: For each of the following, find the critical points using derivative techniques and identify them as maxima or minima. Find the absolute maximum and minimum values of each function if it exists. Identify the intervals on which the function is increasing or decreasing.

1. $f(x) = 4x^3 + 3x^2 - 6x + 1$ on the interval [-2, 1]
2. $f(x) = \sin x - \cos x$ on the interval [0, π]
3. $f(x) = \sqrt[5]{x^2}$ on the interval [-1, 32]
4. $f(x) = x - \ln x$ on the interval [0.1, 5]
5. $f(x) = x + \frac{32}{x^2}$ over all real numbers
6. $f(x) = 2x - e^x$ on the interval [-2, 4]
7. $f(x) = 3x\sqrt[3]{x} - 2x$ on the interval [0, 3]
8. $f(x) = \frac{x^4 + 1}{x^2}$

9. A student drew an *f'* line in order to analyze the behavior of the graph of the function *f(x)*. Draw a graph to match the student's analyses.

a. *f'* _____+__|__-__|__+__|___+____
-1 1 3

b. *f'* ___+___|___+___|___-___|___-___
-1 1 3

10. Sketch graphs to match the descriptions below.

a. $f'(x)$ is always equal to 1. When $x = 0$, $f(x) = 3$.
b. $f(-1) = 2$ and $f'(x)$ is negative when $x < -1$, zero when $x = -1$ and positive when $x > -1$
c. $f'(x)$ is positive when $x < 3$, and negative when $x > 3$. At $x = 3$, $f'(x)$ is undefined and $f(x) = 4$.
d. $f'(x)$ is positive everywhere except at x = 0. When x = 0, $f'(x) = 0$ and $f(x) = 2$.
e. $f'(x)$ is negative everywhere except at x = 1. $f'(1)$ is undefined and $f(1) = 3$.
f. $f'(x) = 0$ at x = -2 and x = 4. When $x = -2$, $f'(x) > 0$. For $-2 < x < 4$, $f'(x) < 0$. $f(-2) = 3$ and $f(4) = 2$.

11. A function *f* is differentiable on the interval [-2, 2]. The table gives the values of *f'* for selected values of *x*. Sketch the graph of *f*, approximate the critical numbers and identify the relative extrema.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
f '(x)	-5	-3	-0.5	0.5	4	3	-0.5	-6	-10

In exercises 12 and 13, the derivative of the function $y = f(x)$ is given. At what points, if any, does the graph have a relative minimum or relative maximum?

12. $\dfrac{dy}{dx} = (x-1)^2 (x-2)$

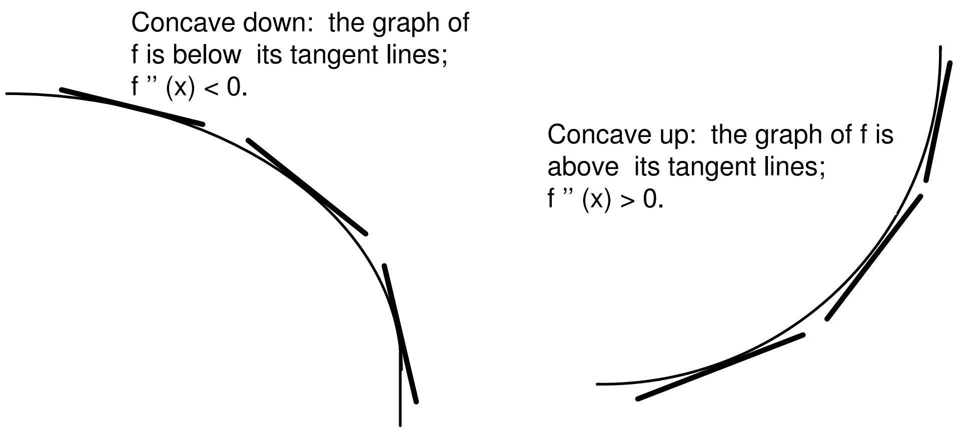
13. $\dfrac{dy}{dx} = (x-1)^2 (x-2)(x-4)$

True or False?

14. If $f(x) = ax^3 + b$ and f is increasing on (-1, 1), then $a > 0$.
15. Every n^{th}-degree polynomial has (n - 1) critical numbers.
16. A continuous function defined on a closed interval must attain a maximum value on that interval.
17. It is possible for a function to have an infinite number of critical points.
18. If $f'(x) > 0$ for all x on an interval, then f is increasing on that interval.

Part 2: Curve Sketching and Inflection Points

The first derivative, $f'(x)$ tells us the rate of change of the function $f(x)$. Similarly, the second derivative $f''(x)$ tells us the rate of change of $f'(x)$. So, if the first derivative tells us if the function is increasing or decreasing, the second derivative tells us where the graph is curving upward and where it is curving downward. If a graph is curving up from its tangent lines, the first derivative is increasing ($f''(x) > 0$) and the graph is said to be concave up. If a graph is curving down from its tangent lines, the first derivative is decreasing ($f''(x) < 0$) and the graph is said to be concave down.



Example 1: Determine the open intervals of the graph $f(x) = x^3 - 9x^2 - 81x + 32$ where $f(x)$ is concave upward or concave downward.

Solution: First, find the second derivative of the function.

$$f'(x) = 3x^2 - 18x - 81$$
$$f''(x) = 6x - 18 = 6(x-3)$$

To find where f' is increasing, find where f'' is positive. To find where f' is decreasing, find where f'' is negative. The easiest way to do this is to set up a number line. Find where $f''(x) = 0$ by setting the second derivative equal to zero and solving the equation: $f''(x) = 0$ when $6(x-3) = 0$ or when $x = 3$.

Since f'' is a polynomial function, it is continuous so f'' cannot change sign except at this point. To the left of 3, the sign of f'' is negative. To the right of 3, the sign of f'' is positive.

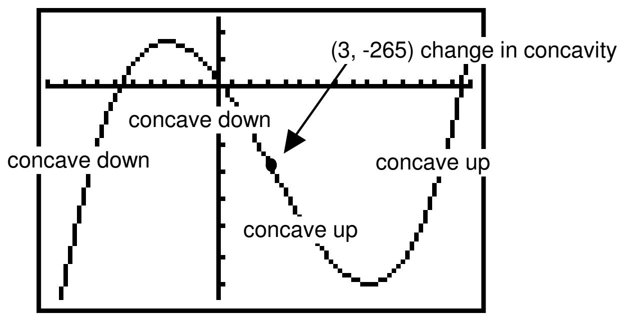
This can be summarized on a number line.

f'' __-__|___+___
3

So, $f'(x)$ is decreasing to the left of 3; i.e., *f(x)* is concave down to the left of 3. $f'(x)$ is increasing to the right of 3; i.e., *f(x)* is concave up to the right of 3. Evaluating the function at 3 , we find $f(3) = -2655$.

To answer the question posed by the problem, *f(x)* is concave down on the interval $(-\infty, 3)$ and concave up on the interval $(3, \infty)$.

This is the same function we examined in Example 1 in Part 1. Study the graph again.



The point (3, -265) is called an inflection point.

A point at which the graph of a function changes concavity is called an inflection point of f.

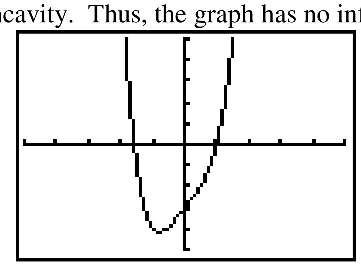
An inflection point may occur where f '' (x) = 0 or where f '' (x) is undefined.

Consider the last statement and the following example.

Example 2: Determine the open intervals where the graph of $f(x) = x^4 + 2x - 3$ is concave upward or concave downward.

Solution: $f'(x) = 4x^3 + 2$ and $f''(x) = 12x^2$

Note that $f''(x) = 0$ when $x = 0$. But also note that $f''(x)$ is always positive; i.e., it is always concave up. There is never a change in concavity. Thus, the graph has no inflection point.



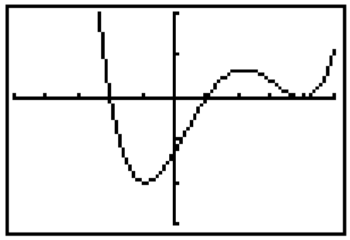
[-5 ,5] x [-5, 5]

Homework Part 2: Curve Sketching and Inflection Points

1. What is the second derivative of a linear function? What does this indicate about the concavity of the graph?

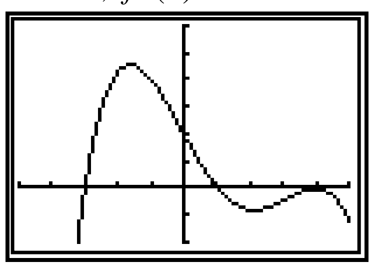
2. What is the second derivative of a quadratic function? What does this say about the concavity of the graph?

3. The graph below is the graph of the derivative, $f'(x)$. What x values are inflection points for $f(x)$?



[-5, 5] x [-15, 10]

4. The graph below is the second derivative, $f''(x)$. What x values are inflection points for $f(x)$?



[-5, 5] x [-10, 30]

Find all relative extrema, points of inflection and intervals where the functions are concave up or concave down for the following functions using derivative techniques.

5. $f(x) = x^3 - 12x$

6. $f(x) = \frac{1}{4}x^4 - 2x^3$

7. $f(x) = x - \sin x$ for the interval [0, 2π]

8. $f(x) = \sin x + \cos x$ for the interval [0, 2π]

9. $f(x) = 2 + 3\cos x$ for the interval [0, 3π]

10. $f(x) = e^x - \ln x^2$

11. $f(x) = x^2 - \frac{2}{x}$

12. $f(x) = 2\sqrt[4]{x}$

13. $f(x) = x + \frac{1}{x}$

14. $f(x) = 3\sqrt[3]{x} - 2x$ for the interval [0, 2π]

15. Sketch a continuous curve $y = f(x)$ with the following properties:

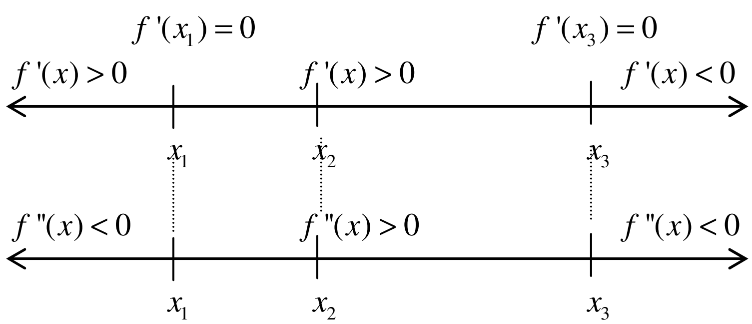
x	$f(x)$	Graph
$x < 3$		decreasing, concave up
3	1	horizontal tangent
$3 < x < 6$		increasing, concave up
6	5	inflection point
$6 < x < 10$		increasing, concave down
10	8	horizontal tangent
$x > 10$		decreasing, concave down

16. Sketch a continuous curve $y = f(x)$ having the following properties:

$$f(-3) = 5$$
$$f(0) = 3$$
$$f(3) = 0$$
$$f'(x) > 0 \quad for \quad |x| > 3$$
$$f'(3) = f'(-3) = 0$$
$$f'(x) < 0 \quad for \quad |x| < 3$$
$$f''(x) < 0 \quad for \quad x < 0$$
$$f''(x) > 0 \quad for \quad x > 0$$

For problems 17 and 18, sketch a possible graph of $y = f(x)$ using the given information about the derivatives $f'(x)$ and $f''(x)$.

17.



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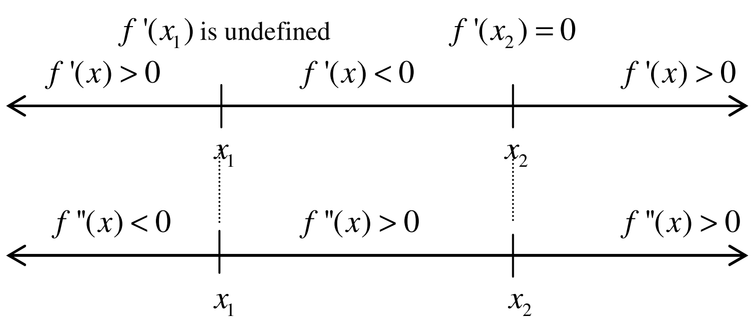


19. Find a, b, c, and d such that the cubic polynomial function $f(x) = ax^3 + bx^2 + cx + d$ satisfies the given conditions: relative maximum is (3, 3); relative minimum is (5, 1) and inflection point is (4, 2). (Hint: Use matrices to solve the system.)

True/False

20. The graph of every cubic polynomial has precisely one point of inflection.

21. The graph of $f(x) = \dfrac{1}{x}$ is concave down for $x < 0$ and concave up for $x > 0$, and thus it has an inflection point at $x = 0$.

22. If $f(x) = 3x^6 + 2x^4 + 3x^2$ then the graph of f is concave up over all real numbers.

23. The graph of $y = \sin x$ has infinitely many inflection points.

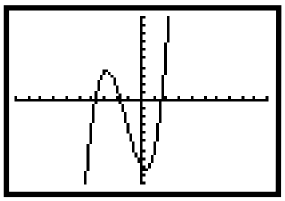
24. If $f''(x) = 0$, then f has an inflection point at $(c, f(c))$.

25. A quadratic function has no inflection points.

Match the graph of f(x) in Column A with the graph of its derivative f' (x) in Column B and the graph of its second derivative in Column C. All graphs are plotted in the standard viewing window.

Column A: *f(x)* | Column B: *f ' (x)* | Column C: *f '' (x)*

26.



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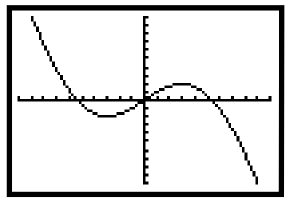

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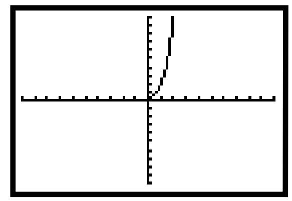

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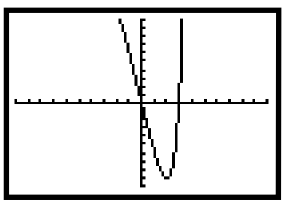

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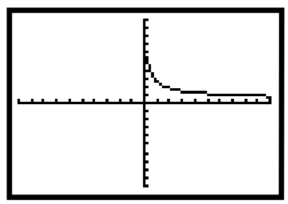

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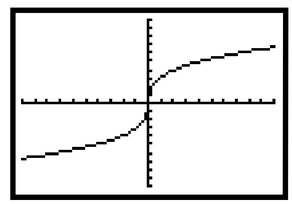

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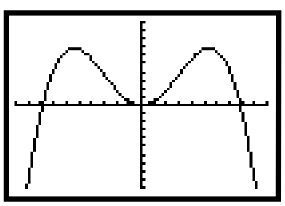

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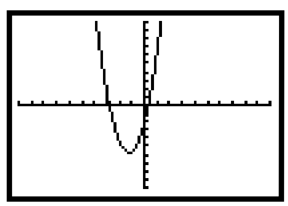

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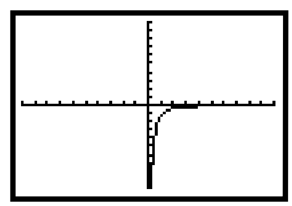

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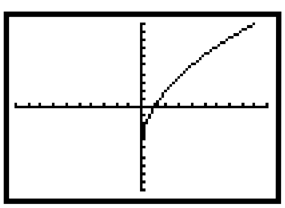

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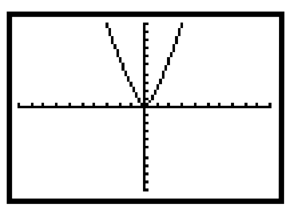
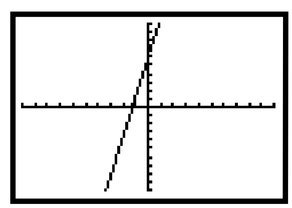


I.



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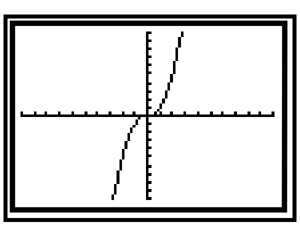

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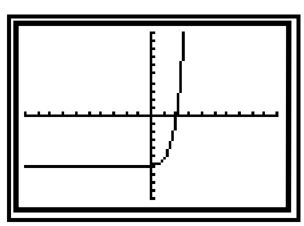

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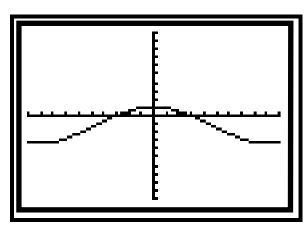


Part 3: Optimization Problems

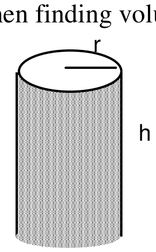
In many real life problems, it is often of interest to find the greatest or least value of a function. The other relative extrema are of no interest. Engineers may want to find the strongest support when constructing a building. Biologists may want to find the lowest temperature at which a certain type of organism can survive. City planners want to design traffic patterns that allow for a maximum traffic flow. There is a certain procedure to follow when solving this type of problem:

1. Assign variables to all given quantities to be determined. When feasible, draw a sketch and label the picture with the quantities.
2. Find a primary expression for the quantity to be optimized that is maximized or minimized.
3. Reduce this primary equation to one having a single independent variable. This may involve using secondary equations (restrictions) relating the variables of the original equation.
4. Determine the possible domain for the independent variable, usually an interval. Evaluate the primary equation at these endpoint values.
5. Find the derivative.
6. Using the derivative, find the critical points (A) singular points **f' is undefined**, B) stationary points **f'=0**, and C) endpoints **domain**).
7. Justify the optimal solution by either A) making a first derivative number line or B) using the second derivative test.
8. Find a solution to the problem and answer the question that was asked.

Example 1: What are the dimensions of an aluminum can that can hold 16 ounces of soda and that uses the least material? Assume that the can is cylindrical, and is capped on both ends. 16 ounces ~ 470 ml. Do your computations in metric units.

Solution:

1. The two variables necessary when finding volume and surface area of a cylinder are radius and height.



2. The primary expression to be minimized is surface area. Surface Area = $2\pi rh + 2\pi r^2$.
3. The formula for volume of a cylinder is $\pi r^2 h$. In this case, the volume is 470, so 470 = $\pi r^2 h$.

   Solving for h, we get $h = \dfrac{470}{\pi r^2}$. Substituting this expression into our primary equation, we get

   Surface Area =. $2\pi r\left(\dfrac{470}{\pi r^2}\right) + 2\pi r^2$ Simplifying the equation, we obtain the function we wish to

   minimize: $A(r) = \dfrac{940}{r} + 2\pi r^2$

4. The domain of this function is $r > 0$, so there are no endpoints to test.

5. $A'(r) = 4\pi r - \dfrac{940}{r^2}$.

6. A) The only singular point is at r = 0. This is not part of the domain and thus can be disregarded.

6. B) Setting the derivative equal to zero in order to find the stationary point(s), we get

$$4\pi r - \frac{940}{r^2} = 0$$

$$4\pi r = \frac{940}{r^2}$$

$$\pi r^3 = 235$$

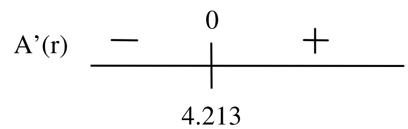
$$r^3 = \frac{235}{\pi}$$

$$r \approx 4.213 \ cm.$$

6. C) There are no endpoints for the domain

7. Justify by

A) using a first derivative number line



Because A' is negative (the area function is decreasing), then A' is zero at r=4.213, and then A' is positive (the area function is increasing) we can conclude that A(4.213) is a minimum.

OR

B) using the second derivative test.

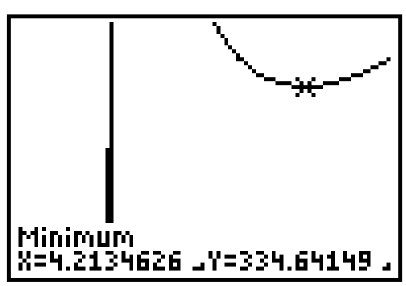
$$A''(r) = 4\pi + \frac{1880}{r^3}$$

$$A''(4.213) > 0$$

Because the second derivative is positive when evaluated at the critical point r=4.213, the area function is concave up and A(4.213) must be a minimum.

8. Substituting this value into $h = \frac{470}{\pi r^2}$, we get $h \approx 8.427 \ cm.$ Therefore, the minimum Surface Area $\approx 334.641 \ sq. \ cm.$ when the dimensions are $r \approx 4.213 \ cm.$ and $h \approx 8.427 \ cm.$

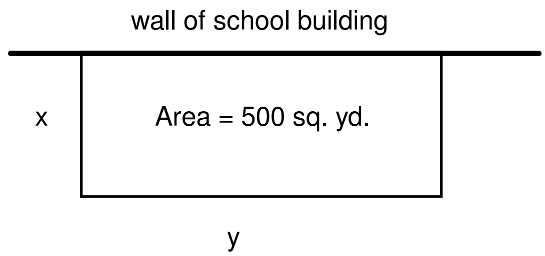
You can confirm your findings with a graph.



[-2, 6] x [-2, 450]

Example 2: A group of parents are planning a children's playground. In order to save on fencing, they are positioning the playground so that one side is immediately adjacent to the school building, as shown below. If the playground is to be a rectangle 500 square yards in area, what dimensions should they use if they want to use the least amount of fencing?

Solution: 1. Let x be the width of the playground and y the length of the playground.



2. The amount of fencing necessary is $2x + y$.

3. The area of the rectangle is $500 = xy$. Solving for y, we get $y = \frac{500}{x}$. So the function to be minimized is $F(x) = 2x + \frac{500}{x}$.

4. The domain of this function is $x > 0$.

5. $F'(x) = 2 - \frac{500}{x^2}$.

6. A) The only singular point occurs at $x = 0$ which is not in the domain, so it can be discarded.

   B) Find the stationary points by setting $F'(x) = 0$ and solving, we get

$$2 - \frac{500}{x^2} = 0 \quad \rightarrow \quad 2 = \frac{500}{x^2}$$

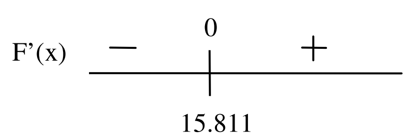
$$x^2 = 250 \quad \rightarrow \quad x = \pm 5\sqrt{10} \approx \pm 15.811$$

   Since -15.811 is not in the domain, it can be discarded.

   C) There are no endpoints to be considered.

7. Justify by

   A) using a first derivative number line



   Because F' is negative (the perimeter function is decreasing), then F' is zero at x=15.811, and then F' is positive (the perimeter function is increasing) we can conclude that F(15.811) is a minimum.

OR

   B) using the second derivative test. $F''(x) = \frac{1000}{x^3}$ *and* $F''(15.811) > 0$

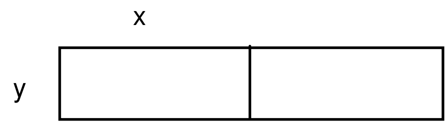
   Because the second derivative is positive when evaluated at the critical point x=15.811, the area function is concave up and F(15.811) must be a minimum.

8. Thus, the playground size that minimizes the amount of fencing is 15.811 yd. x 31.622 yd.

Homework Part 3: Optimization Problems

Directions: Solve each of the following using calculus methods.

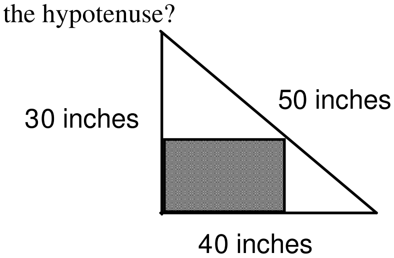
1. A farmer wishes to fence off two identical adjoining rectangular pens, each with 800 square feet of area as shown below. What are x and y so that the least amount of fence is required?



2. Suppose that the outer boundary of the pens in the previous problem requires heavy fencing that costs $2.50 per foot, but that the middle partition requires fence costing only $1.10 per foot. What dimensions will produce the least expensive fence?

3. Mr. Berry wishes to organize all the pens and markers in his classroom by making several boxes to hold them. He has five sheets of cardboard 9 inches wide and 12 inches long with which he will make five open-topped boxes. How can he use them to make boxes with the greatest possible volume?

4. Mrs. Eagen wishes to construct a storage box in the corner of her attic. The corner of the attic is triangular shaped, with sides of 30 inches, 40 inches and 50 inches, as shown below. What is the biggest storage box that could be constructed in the attic, if she wants the box to be a rectangle with two sides along the two shorter sides of the attic corner and one corner on the hypotenuse?



5. Mr. Stueben wishes to construct a fish tank with a volume of 3 cubic meters and would like the base of the tank to be a rectangle twice as long as it is wide. The base and sides of the tank are to be made of glass. What dimensions of the tank will use the least amount of glass?

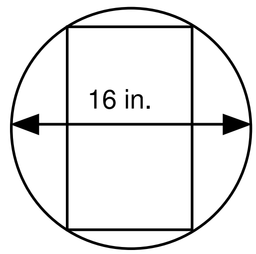
6. Some airlines have restrictions on the size of items of luggage that passengers are allowed to take with them. One has a rule that the sum of the length, wide and height of any piece of luggage must be less than 60 inches. A passenger wants to get a piece of luggage of the maximum-allowable volume. If the length and width are to be equal, what should the dimensions be and what will be the volume? If the length is to be twice the width, what should the dimensions be?

7. The graphs of $y = \sqrt{x}$, x = 8 and y = 0 bound a region in the first quadrant. Find the dimensions of the rectangle of maximum area that can be inscribed in this region (the sides of the rectangle should be parallel to the axes).

8. What values of a and b cause $f(x) = x^3 + ax^2 + bx$ to have a relative maximum at x = -1 and a relative minimum at x = 3?

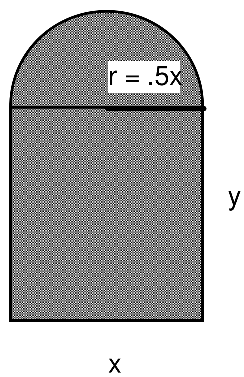
9. Mrs. Parnell bought some land in Winchester and is planning to plant a small apple orchard. She learns that if she plants up to 60 trees on her plot of land, the average harvest from each tree will be about 200 pounds, but for each additional tree planted the yield will go down by an average of 1.5 pounds per tree, as a result of overcrowding. Naturally she wants to plan for the maximum yield of apples. How many trees should she plant?

10. The strength of a wooden beam with a rectangular cross-section is proportional to the product of its width and the square of its depth. What are the width and depth of the strongest beam that can be cut from a log with a circular cross-section of diameter 16 inches?



11. Find two positive numbers whose sum is 4, such that the sum of the cube of the first and the square of the second is as small as possible.

12. An ornamental window is constructed by adjoining a semi-circle to the top of an ordinary rectangular window, as shown below. Find the dimension of the window of maximum area if the total perimeter is 20 feet.



13. A sports center is to be constructed. It consists of a rectangular region with a semicircle on each end. If the perimeter of the room is to be a 500-meter running track, find the dimensions that will make the area as large as possible.

14. A flyer is to contain 20 square inches of print. The margins on each side are 1.5 inches. Find the dimension of the page so that the least paper is used.

15. A retailer has determined that the cost C for ordering and storing x units of a product is $C = 4x + \frac{400,000}{x}$ for $1 \le x \le 400$. The delivery van can bring at most 400 units per order. Find the order size that will minimize the cost.

16. The formula for the power output P of a battery is $P = VI - RI^2$ where V is the electromotive force in volts, R is the resistance in ohms, and I is the current in amperes. Find the current that corresponds to a maximum value of P in a battery for which V = 9 volts and R = 0.3 ohm. Assume that a 12 amp fuse bounds the output in the interval $0 \le I \le 12$.

17. The crankshaft of an engine rotates at a constant rate of 300 revolutions per minute. The horizontal velocity (cm/min) of a point on the crankshaft is given by the equation $v = -600\pi \sin\theta$ where θ is the central angle of the crankshaft. What values of θ produce a maximum horizontal velocity?

18. A salt container is to be made in the shape of a right circular cylinder. If it is to contain 1000 cubic centimeters of salt, find the dimensions for the container that requires the least amount of material.

Part 4: Other Applications

There are many real world situations where it is useful to know intervals where the function is increasing or decreasing. A common application of differential calculus involves particle motion problems along a horizontal line. It is possible to determine when the particle is moving left, moving right, when it is speeding up and when it is slowing down.

Example: AP Calculus Problem 1975 AB 2: A particle is moving along the x-axis in such a way that its position at time t for $t \geq 0$ is given by $x(t) = \frac{1}{3}t^3 - 3t^2 + 8t$.

a. Show that at time $t = 0$ the particle is moving to the right.
b. Find all values of t for which the particle is moving to the left.
c. What is the position of the particle at time $t = 3$?
d. When $t = 3$, what is the total distance the particle has traveled?
e. Find the displacement of the particle on the interval $t \in [0,3]$
f. When $t = 3$, what is the acceleration of the particle?
g. When is the particle accelerating (speeding up), decelerating (slowing down)?

Solution: When the first derivative is positive, the particle moves to the right. When the first derivative is negative, the particle moves to the left. When the first derivative equals zero, the particle comes to a stop.

$x'(t) = t^2 - 6t + 8 = (t-4)(t-2)$. So $x'(t) = 0$ when $t = 4$ or $t = 2$.

$x'(t)$ |______+_____|_____-_____|_____+_____
0 2 4

a. At $t = 0$, $x'(0) = (0-4)(0-2) = 8$. Since $x'(0)$ is positive, the particle is moving to the right.

b. The particle is moving to the left on the interval (2, 4)

c. The position of the particle at time $t = 3$ is $\left(\frac{1}{3}\right)(3^3) - 3(3^2) + 8(3) = 9 - 27 + 24 = 6$.

d. The particle begins at position $x(0) = 0$. It moves right until $t = 2$ and the position at $t = 2 \quad is \quad \frac{20}{3}$. It then moves left. At $t = 3$, it has reached a position of 6. Therefore, the total distance traveled is $\frac{20}{3} + \left(\frac{20}{3} - 6\right) = \frac{20}{3} + \left(\frac{2}{3}\right) = \frac{22}{3} = 7\frac{1}{3}$ units.

e. The displacement of the particle is the net change in position from time 0 to time 3. During this time the particle moves from position 0 to position 6 yielding a displacement of 6 units.

f. The acceleration is determined by the second derivative. $x''(t) = 2t - 6$. At $t = 3$, the acceleration is 0.

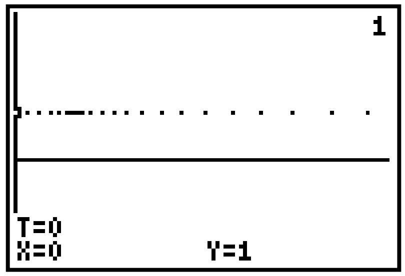
g. The particle is accelerating when both the velocity and acceleration are positive or both are negative.

$x'(t) = t^2 - 6t + 8 = (t-4)(t-2)$ $\qquad v(t) = x'(t)$ $\quad$ + 0 − − 0 +

$x''(t) = 2t - 6 = 2(t-3)$ $\qquad a(t) = x''(t)$ $\quad$ − | − 0 + | +

2 3 4

It is possible to examine the motion of a particle in parametric mode. Enter the function in X1T and let Y1T = 1 so that the motion is off the axis and easier to view. Set the mode to Dot. Use the following Window Settings: Tmin = 0, Tmax = 20, Tstep = 0.2, Xmin = 0, Xmax = 40, Xscl = 0, Ymin = -2, Ymax = 3, and Yscl = 0. Graph the function and then user the TRACE key. As you push the right arrow key, you will trace the motion of the particle.



Homework Part 4: Other Applications

Directions: Use differential calculus methods to solve the following problems. (Let $t \geq 0$)

1. A particle is moving on the x-axis. Its position is given by $x(t) = 2t^3 - 13t^2 + 22t - 2$. Find the intervals when the particle is moving right, when it is moving left, when its acceleration is positive, and when its acceleration is negative, when the particle is speeding up and slowing down.

2. The position of a particle along a horizontal number line at time t is given by the function $x(t) = -t^2 + 6t - 8$.
   a. What is the largest time interval for which x is an increasing function? In which direction is the motion during this time?
   b. At what time does the particle change direction?
   c. On what time interval is the particle slowing down?

3. Two particles are moving along a coordinate line. At the end of t seconds their distances from the origin, in feet, are given by $x = 4t - 3t^2$ and $y = t^2 - 2t$, respectively.
   a. When do they have the same velocity?
   b. When do they have the same speed? (The speed of a particle is the absolute value of its velocity.)
   c. When do they have the same position?

4. After birth, an infant normally will lose weight for a few days and then start gaining. A model for the average weight W (in pounds) of infants over the first 4 weeks following birth is $W = 0.033t^2 - 0.402t + 7.210$, $0 \leq t \leq 28$ where t is measured in days. Find the intervals on which W is increasing or decreasing.

5. The profit P (in dollars) made by a fast food restaurant selling x tacos is $P = 1.09x - \dfrac{x^2}{40,000} - 10,000$ for $0 \leq x \leq 50,000$. Find the intervals where P is increasing and where it is decreasing.

Homework Review Exercises

For problems 1 - 5, find the critical points (if any) and the intervals on which the function is increasing and decreasing. Also, find any points of inflection and the intervals on which the function is concave up and concave down.

1. $f(x) = (x-3)\bullet\sqrt{x}$
2. $f(x) = 2x + 2\cos x$ for $[0, 2\pi]$
3. $f(x) = x^3 + x + \frac{4}{x}$
4. $f(x) = x^4 - 12x^3 + 48x^2 - 64x$
5. $f(x) = 2x\sqrt[3]{x^2} - 5x\sqrt[3]{x}$

6. A rectangular page of print is to contain 37 square inches of print. The margins at the top and bottom of the page are each 1.5 inches. The margins on each side are 1 inch. What should the dimensions of the page be so the least amount of paper is used?

7. A farmer plans to fence in a rectangular field adjacent to a river. The field must contain 150,000 square yards in order to provide enough grass for his herd of cattle. What dimensions would require the least amount of fencing if no fencing is needed along the river?

8. A particle moves along a horizontal line in such a way that its position at time t is given by $x(t) = t^3 - 12t^2 + 36t - 10$ where x is measured in feet and t in seconds.

   a. When is the velocity 0?
   b. When is the velocity positive?
   c. When is the point moving backward?
   d. When is the acceleration positive?
   e. When is the particle slowing down?

9. AP Problem 1986 AB 1: Let f be the function defined by the function $f(x) = 7 - 15x + 9x^2 - x^3$ for all real numbers x.

   a. Find the zeros of f.
   b. Write an equation of the line tangent to the graph of f at $x = 2$.
   c. Find the x-coordinates of all points of inflection of f. Justify your answer analytically.

10. Sketch a possible graph for a function that is continuous for all x and for which

$$f(-2) = -f(2) = -4$$
$$f'(-2) = f'(2) = 0$$
$$f''(-2) > 0$$
$$f''(2) < 0$$

11. The position of a particle is defined by the function $x(t) = \frac{8}{3}t^3 - t^2 - 15t + 4$.

   a. Find the maximum velocity of the particle on the interval [0, 5].
   b. Find the minimum acceleration of the particle on the interval [0, 5].

12. Sketch the graph of a continuous function that satisfies all of the following conditions:

$f(0) = f(3) = 3$	$f'(x) > 0 \quad on \quad (0,2)$	$f''(x) < 0 \quad on \quad (0,3)$
$f(2) = 5,$	$f'(x) < 0 \quad on \quad (2,4)$	$f''(x) < 0 \quad on \quad (4,5)$
$f(4) = 1,$	$f'(x) < 0 \quad on \quad (4,5)$	$f''(x) > 0 \quad on \quad (3,4)$
$f(6) = -1$	$f'(2) = f'(4) = 0$	
	$f'(x) = -1 \quad on \quad (5,6)$	

True/False

13. The sum of two increasing functions is increasing.
14. The maximum value of $y = 2\sin x + 3\cos x$ is 5.
15. If c is a critical number of f, then f has a relative extremum at $x = c$.
16. If the acceleration of an object is negative, then its velocity is decreasing.
17. If $x = 4t^3 + 5t - 200$ gives the position of an object on a horizontal coordinate line at time t, then that object is always moving right.
18. An absolute maximum is always a relative maximum.
19. If $a < c < b$ and c is a critical number of f, then f has a relative extremum in (a, b).