

DERIVATIVES: RULES

Part 1: Derivatives of Polynomial Functions

We can use the definition of the derivative in order to generalize solutions and develop rules to find derivatives. The simplest derivatives to find are those of polynomial functions.

Example 1: Find the derivative of the constant function $f(x) = c$ using the definition of derivative.

Solution:
$$\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{c - c}{\Delta x} = 0$$

So, the derivative of a constant is 0. This corresponds to the graphing of derivatives we did earlier. The graph of a constant function is a horizontal line and the slope of a horizontal line is 0.



$$f(x) = c$$

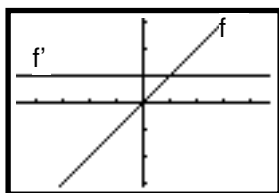
$$f'(x) = 0$$

Symbolically, we write

Constant Rule: If $f(x) = c$, then $f'(x) = 0$.

Example 2: Find the derivative of each of the following functions based on their functions. The function and its derivative are pictured in each diagram. A table of values for f and f' is also shown. Study the values carefully. The function is given at the right.

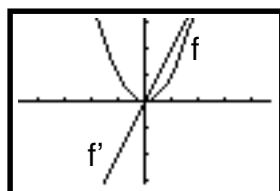
Complete the list of derivatives



X	Y1	Y2
-3	-3	1
-2	-2	1
-1	-1	1
0	0	1
1	1	1
2	2	1
3	3	1

X=0

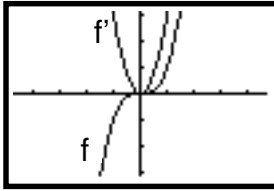
$f(x) = x$
 $f'(x) = \underline{\hspace{2cm}}$



X	Y1	Y2
-3	9	6
-2	4	4
-1	1	2
0	0	0
1	1	2
2	4	4
3	9	6

X=0

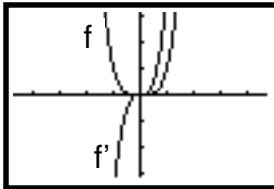
$f(x) = x^2$
 $f'(x) = \underline{\hspace{2cm}}$



X	Y1	Y2
-3	-27	27
-2	-8	12
-1	-1	3
0	0	1E-6
1	1	3
2	8	12
3	27	27

X=0

$$f(x) = x^3 \quad f'(x) = \underline{\hspace{2cm}}$$



X	Y1	Y2
-3	81	-108
-2	16	-32
-1	1	-4
0	0	0
1	1	4
2	16	32
3	81	108

X=0

$$f(x) = x^4 \quad f'(x) = \underline{\hspace{2cm}}$$

A pattern is emerging when we take the derivative of a power. The exponent becomes the coefficient of the derivative and the power of the derivative is one less than the power of the function. This is called the power rule and symbolically it is written as follows.

Power Rule: If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

We noted above that the derivative of $f(x) = x^2$ is $2x$. What is the derivative of $5x^2$? $16x^2$? $1023x^2$? Is there a rule that governs multiplying a function by a constant? It is possible to answer this question by once again going back to the definition of the derivative. The question we are asking is “what is the derivative of kx^2 ”?

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{k(x + \Delta x)^2 - kx^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{kx^2 + 2kx\Delta x + k\Delta x^2 - kx^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (2kx - k\Delta x) \\ &= 2kx \\ &= k(2x) \end{aligned}$$

So, the derivative of a function multiplied by a constant is the constant multiplied by the derivative. Or,

Constant Multiple Rule: The derivative of $kf(x)$ is $kf'(x)$.

What is the derivative of a sum or difference of several powers; i.e., what is the derivative of a polynomial?

Sum Rule: The derivative of a sum $f(x) + g(x)$ is the sum of the derivatives, $f'(x) + g'(x)$.

Likewise, the derivative of a difference is the difference of the derivatives.

Difference Rule: The derivative of a difference $f(x) - g(x)$ is the difference of the derivatives, $f'(x) - g'(x)$.

With these few simple rules, we can now find the derivative of any polynomial.

Example 2: Find the derivative of $f(x) = 5x^6 - 3x^4 + 2x^3 - 8x^2 + 17$

Solution: Find the derivative of each term of the polynomial using the constant multiple rule and power rules. Then, add or subtract the derivative of each term, as appropriate.

The derivative of $5x^6$ is $5(6x^5) = 30x^5$.

The derivative of $-3x^4$ is $-3(4x^3) = -12x^3$.

The derivative of $2x^3$ is $2(3x^2) = 6x^2$.

The derivative of $-8x^2$ is $-8(2x) = 16x$.

The derivative of 17 is 0.

So, the derivative of $f(x) = 5x^6 - 3x^4 + 2x^3 - 8x^2 + 17$ is $f'(x) = 30x^5 - 12x^3 + 6x^2 - 16x$.

Example 3: Find the slope of the tangent line to the curve $f(x) = 3x^2 - 4x + 3$ at $x = 1$.

Solution: Using the derivative rules, $f'(x) = 6x - 4$. At $x = 1$, the derivative is $6(1) - 4 = 2$. This can also be written as $f'(1) = 2$.

Homework Exercises Part 1: Derivatives of Polynomial Functions

Find the derivative of each of the following.

1. $f(x) = 4x^5 - 3x^2$

5. $f(x) = x^6 - 8x$

2. $f(x) = x^3 + x^2 - x + 1$

6. $f(x) = \frac{1}{2}x^4 - \frac{1}{4}x^2 + 2x - 4$

3. $f(x) = 0.1x^{10} - 0.5x^5 - 0.3x^3 - 0.25$

7. $f(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$

4. $f(x) = (3x - 2)^2$

8. $f(x) = (x + 7)(2x - 3)$

Find the indicated derivative.

9. $f'(1)$ if $f(x) = 3x^4 - 5x + 2$

11. $f'(2)$ if $f(x) = 3x^5 - 2x^2 + 4$

10. $f'\left(\frac{1}{2}\right)$ if $f(x) = 5x^3 - 3x^2 + 2x - 6$

12. $f'(-3)$ if $f(x) = 2x^4 + 4x^2 + 6$

Write the equation of the tangent line at the indicated point.

13. $f(x) = x^2$ at $(2, 4)$

15. $f(x) = 4 - x^2$ at $(-1, 3)$

14. $f(x) = (x - 1)^2 + 1$ at $(1, 1)$

16. $f(x) = 2 - (x - 1)^2$ at $(-2, -7)$

17. Use the definition of derivative to prove that the derivative of a linear function is a constant; i.e., prove that if

$$f(x) = ax + b \text{ then } f'(x) = a, \text{ using } \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

18. Use the definition of derivative to prove that the derivative of a quadratic function is a linear function; i.e., prove that if $f(x) = ax^2 + bx + c$, then $f'(x) = 2ax + b$, using $\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$.
19. Find all points on the graph of $y = x^3 - x^2$ where the tangent line is horizontal.
20. Find all points on the graph of $y = \frac{1}{3}x^3 + x^2 - 3x + 1$ where the tangent line has slope 1.
21. The height x in feet of a ball above the ground at t seconds is given by the equation $x = -16t^2 + 40t + 100$
- What is the instantaneous velocity at $t = 2$?
 - When is the instantaneous velocity equal to 0?
22. There are two tangent lines to the curve $y = 4x - x^2$ that go through the point $(2, 5)$. Find the equations of both of them.
23. Suppose $f'(0) = 2$, and $g'(0) = -1$, find
- The derivative of $f(0) + g(0)$
 - The derivative of $2f(0) - 3g(0)$

Part 2: Derivatives of Negative and Fractional Powers

In the last set of exercises, you proved that the derivative of a linear function is a constant function; the derivative of a quadratic function is a linear function; the derivative of a cubic function is a quadratic function. It is possible to continue these proofs for any n . We generalized these derivatives by using the power rule and constant, sum and difference rules. Thus far, we have used the power rule for positive integer powers only. What of negative and fractional powers?

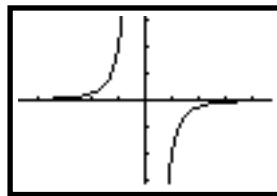
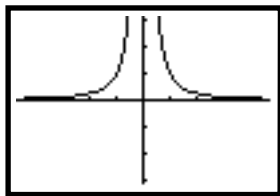
Example 1: Find the derivative of $f(x) = x^{-2} = \frac{1}{x^2}$

Solution: Substituting into the definition, we get

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{(x + \Delta x)^2} - \frac{1}{x^2}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{x^2 + 2x\Delta x + \Delta x^2} - \frac{1}{x^2}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\frac{x^2}{x^2(x^2 + 2x\Delta x + \Delta x^2)} - \frac{x^2 + 2x\Delta x + \Delta x^2}{x^2(x^2 + 2x\Delta x + \Delta x^2)}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^2 - (x^2 + 2x\Delta x + \Delta x^2)}{\Delta x \cdot x^2(x^2 + 2x\Delta x + \Delta x^2)} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-2x\Delta x - \Delta x^2}{\Delta x \cdot x^2(x^2 + 2x\Delta x + \Delta x^2)} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-2x - \Delta x}{x^2(x^2 + 2x\Delta x + \Delta x^2)} = \frac{-2x}{x^4} = \frac{-2}{x^3} = -2x^{-3} \end{aligned}$$

So, the derivative of $f(x) = x^{-2}$ is $f'(x) = -2x^{-3}$. Therefore, the power rule applies to negative exponents.

Study the graphs of $y = x^{-2}$ and $y = -2x^{-3}$ below. Is x^{-2} differentiable at every point?



From the graphs, we see that x^{-2} is not differentiable at $x = 0$.

Is it possible to apply the power rule to fractional exponents? Consider the function $f(x) = \sqrt{x}$.

Example 2: Find the derivative of $f(x) = \sqrt{x} = x^{\frac{1}{2}}$

Solution: Substituting into the definition, we get

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} = \\ &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} = \\ &= \lim_{\Delta x \rightarrow 0} \frac{x + \Delta x - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} = \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} = \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} = \frac{1}{2} x^{-\frac{1}{2}} \end{aligned}$$

So, the derivative of $f(x) = \sqrt{x}$ or $f(x) = x^{\frac{1}{2}}$ is $f'(x) = \frac{1}{2} x^{-\frac{1}{2}}$. Again, the power rule applies.

Homework Exercises Part 2: Derivatives of Negative and Fractional Powers

Find the derivatives of each of the following functions.

1. $f(x) = \frac{1}{x^3}$

4. $f(x) = \sqrt[3]{x}$

2. $f(x) = x^2 - \frac{1}{x^2}$

5. $f(x) = 3x^2 - 4\sqrt{x} - 4$

3. $f(x) = \frac{5}{x^3} - \frac{2}{x^2} + x - \frac{1}{x}$

6. $f(x) = \frac{3x^5 - 4x^3 - 2x^2 - x - 1}{x}$

Find the indicated derivative.

7. $f'(1)$ if $f(x) = \frac{3}{x^4} - \frac{5}{x} + \pi$

9. $f'(32)$ if $f(x) = 3\sqrt[5]{x} - 2\sqrt{x} + 4$

8. $f'\left(\frac{1}{2}\right)$ if $f(x) = \frac{5}{x^3} - \frac{3}{x^2} + \frac{2}{x} - 6$

10. $f'(16)$ if $f(x) = 2\sqrt[4]{x} + 4x^2 + 6$

Write the equation of the tangent line at the indicated point.

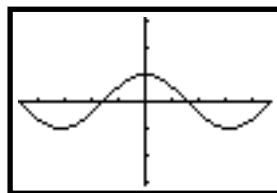
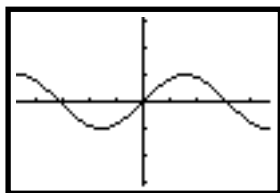
11. $f(x) = \sqrt{x}$ at $(4, 2)$

12. $f(x) = x + \frac{2}{x}$ at $(1, 3)$

13. If $f(x) = 13 - 8x + \sqrt{2}x^2$ and $f'(c) = 4$, find c .

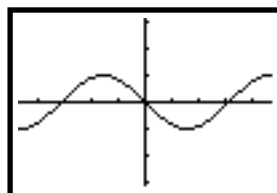
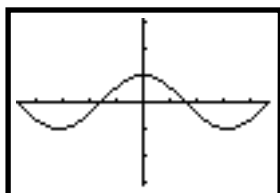
Part 3: Derivatives of the Sine and Cosine Functions

Look at the graph of $f(x) = \sin x$ and its derivative. What is the derivative of $\sin x$?



If $f(x) = \sin x$, then $f'(x) = \cos x$.

Do the same for $f(x) = \cos x$. What is the derivative of $\cos x$?



Be careful! The temptation is to say that the derivative of $f(x) = \cos x$ is $\sin x$, but note that it is $f'(x) = -\sin x$.

If $f(x) = \cos x$, then $f'(x) = -\sin x$.

Example 1: Find the derivative of $f(x) = 2 \sin x - 4 \cos x$.

Solution: The derivative of $2 \sin x$ is $2 \cos x$
 The derivative of $-4 \cos x$ is $-4(-\sin x) = 4 \sin x$.
 So, the derivative of the function is $f'(x) = 2 \cos x + 4 \sin x$.

Example 2: Find the equation of the tangent line to the graph of $y = 3 \sin x$ at $x = \frac{\pi}{3}$

Solution:

The derivative of $f(x) = 3 \sin x$ is $f'(x) = 3 \cos x$.

The value of the derivative at $x = \frac{\pi}{3}$ is $3 \cos\left(\frac{\pi}{3}\right) = 3\left(\frac{1}{2}\right) = \frac{3}{2}$

The y value at $x = \frac{\pi}{3}$ is $3 \sin\left(\frac{\pi}{3}\right) = 3\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2}$

So, the tangent line is $y - \frac{3\sqrt{3}}{2} = \frac{3}{2}\left(x - \frac{\pi}{3}\right)$

Homework Exercises Part 3: Derivatives of the Sine and Cosine Functions

Find the derivative of each of the following functions.

1. $f(x) = 3 \sin x + 6 \cos x$

4. $f(x) = 2x - 2 \cos x$

2. $f(x) = \frac{1}{x} - \sin x$

5. $f(x) = \cos x - \sqrt{x}$

3. $f(x) = \frac{1}{2} \sin x - \frac{2}{x^3}$

6. $f(x) = \sin^2 x + \cos^2 x$

7. Find the equation of the tangent line to $y = \sin x$ at $x = 1$.

8. At time t seconds, the center of a bobbing buoy is $\frac{1}{2} \sin t$ meters above or below water level. What is the velocity of the cork at $t = 0, \pi/2$ and π ?

9. A weight is hanging from a spring. It is compressed 5 cm above its rest position ($x = 0$) and released at time $t = 0$ seconds to bob up and down. Its position at any later time t is $x = 5 \cos t$. What is its velocity at time t ?

You can simulate the up and down motion of the spring y using parametric equations. Graph simultaneously

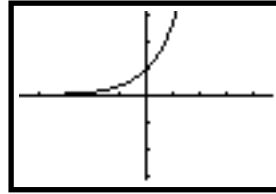
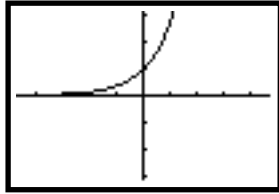
$$\begin{array}{lll} X1(T) = -1 & Y1(T) = 5 \cos t & \\ X2(T) = t & Y2(T) = -5 \sin t & \text{for } 0 \leq T \leq 3\pi. \end{array}$$

Use the TRACE key on your calculator to explore the position and velocity functions. If you change $X1(T) = -1$ to $X1(T) = T$, you can see the up and down motion of the spring.

10. A normal line is perpendicular to the tangent line. Find the equations for the lines that are tangent and normal to the curve $y = \sqrt{2} \cos x$ at the point $\left(\frac{\pi}{4}, 1\right)$.

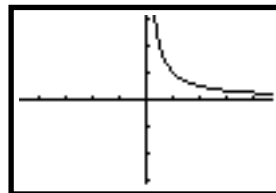
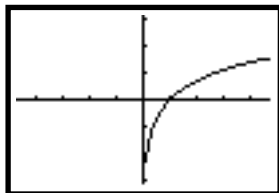
Part 4: Derivatives of the Natural Exponential and Logarithmic Functions

Look at the graph of $f(x) = e^x$ and its derivative. What is the derivative of e^x ?



If $f(x) = e^x$, then $f'(x) = e^x$. The natural exponential function is the only function that has itself as its derivative.

Do the same for $f(x) = \ln x$. What is the derivative of $\ln x$?



If $f(x) = \ln x$, then $f'(x) = 1/x, x > 0$.

It is possible to use the laws of logarithms to aid in finding derivatives of natural logarithmic functions.

Example 1: Find the derivative of $f(x) = \ln x^2$.

Solution: We know that $\log a^b = b \log a$. So, $\ln x^2 = 2 \ln x$ (when $x > 0$). Now, we can apply the constant multiple rule and find the derivative of $f(x)$: $f'(x) = 2\left(\frac{1}{x}\right) = \frac{2}{x}$.

Example 2: Find the derivative of $f(x) = \ln(x^2 \cdot e^x)$

Solution: Again, apply a log rule: $\ln(ab) = \ln a + \ln b$.
So, $\ln(x^2 \cdot e^x) = \ln x^2 + \ln e^x = 2 \ln x + x \ln e$. What is $\ln e$? Recall that the natural logarithmic function and natural exponential functions are inverses of each other. Therefore, $\ln e = 1$. So, $2 \ln x + x \ln e = 2 \ln x + x$. Now, that we have simplified the function, finding the derivative is simple. $f'(x) = 2\left(\frac{1}{x}\right) + 1 = \frac{2}{x} + 1$.

Homework Exercises Part 4: Derivatives of the Natural Exponential & Logarithmic Fns.

Find the derivatives of each of the following.

1. $f(x) = 5x^2 + 4e^x$

6. $f(x) = x^3 - \ln x$

2. $f(x) = (\ln 2)x^2 - (\ln 3)e^x$

7. $f(x) = \pi e^x + \sqrt{x}$

3. $f(x) = \ln x - \frac{1}{x}$

8. $f(x) = \frac{1}{\sqrt[5]{x^2}} - 3 \ln x - 3e^x$

4. $f(x) = \ln x^3$

9. $f(x) = \ln(3x^4 e^x)$

5. $f(x) = \ln\left(\frac{2x}{3e^x}\right)$

10. Find the equation of the tangent line to the graph of the function at the point (1, 0).

a. $f(x) = \ln x^{\frac{3}{2}}$ b. $f(x) = \ln x^2$

11. Consider the function $f(x) = 1 - e^x$. a. Find the slope of $f(x)$ at the point where it crosses the x-axis.

b. Find the equation of the tangent line to the curve at this point.

c. Find the equation of the normal line to the curve at this point.

12. Find the equation of the tangent line to the function $f(x) = 3x^2 - \ln x$ at the point (1, 3).

Part 5: Differentiability

When we examined functions with negative exponents as well as the natural exponential and logarithmic functions, we found that there were certain values where we could not find the derivative of a function. There are actually several situations that destroy differentiability at a point. In order to determine whether a function is differentiable at a given point, we will again return to the definition of the derivative.

Example 1: Consider the function $f(x) = |x - 1|$. Find the derivative of this function at $x = 1$.

Solution: Substituting into the definition, we get

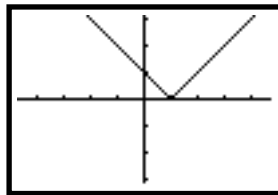
$$\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{|1 + \Delta x - 1| - |1 - 1|}{\Delta x} =$$
$$\lim_{\Delta x \rightarrow 0} \frac{|\Delta x|}{\Delta x}$$

The limit does not exist, because the limit from the left and from the right differ:

$$\lim_{\Delta x \rightarrow 0^+} \frac{|\Delta x|}{\Delta x} = 1 \quad \text{but} \quad \lim_{\Delta x \rightarrow 0^-} \frac{|\Delta x|}{\Delta x} = -1$$

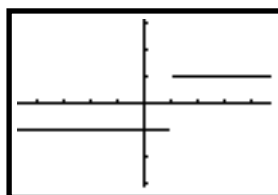
Therefore, the derivative does not exist at $x = 1$. Or, in other words, the function $f(x) = |x - 1|$ is not differentiable at $x = 1$.

Examine the graph of the function at $x = 1$.

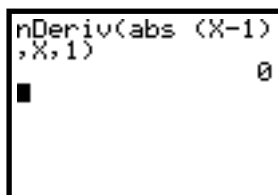


The graph comes to a sharp point at $x = 1$.

Now, study the graph of the derivative as given by the calculator. It supports the results found by using the definition of the derivative.



Now, use the `nDeriv` command on your calculator to find the derivative of the function at $x = 1$.



Obviously, the calculator is giving an incorrect answer. Why is that? Recall that your calculator uses the symmetric difference quotient when computing the derivative. Essentially, it is averaging the right and left hand limits as it approaches the given value. In this case, the left and right hand limits are -1 and 1 , and when averaged they give 0 . No matter how small a Δx you choose, you will always get left and right hand limits of -1 and 1 and you will get an averaged value of 0 . The calculator will always give an incorrect answer when the graph comes to a sharp point, so it is important to examine the graph of a function when using the calculator to find the numeric derivative.

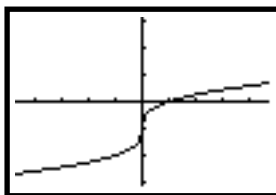
Note, the derivative of $f(x)$ does exist at other points. You can find $f'(0)$. It is -1 . And, $f'(10) = 1$.

Example 2: Consider the function $f(x) = \sqrt[3]{x} - 1$. Find the derivative of this function at $x = 0$.

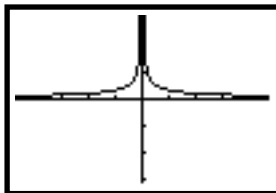
Solution: Using the power rule, we know $f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{x^2}}$

Obviously, when we try to evaluate this at $x = 0$, we get an undefined value. What does this mean?

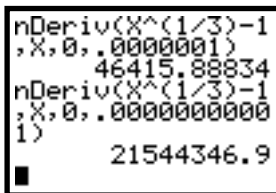
Study the graph of the function.



At $x = 0$, the graph becomes very steep. When you examine the graph of the derivative you find an asymptote at $x = 0$:



What does your calculator tell you? There are two values displayed below, one for a Δx of 0.0000001 and one for a Δx of 0.0000000001. Note that the values are getting very large; they are approaching infinity.

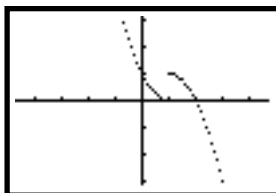


This means that the function has a vertical tangent line at the point $x = 0$, a tangent line with an undefined slope. Again, you cannot trust your calculator to find the correct derivative for you at this particular value.

It is possible to find the derivative of this function at any other value on your calculator or by using the power rule.

Example 3: Consider the piecewise function $f(x) = \begin{cases} x^2 - 2x + 1 & \text{if } x < 1 \\ 2x - x^2 & \text{if } x \geq 1 \end{cases}$ and find the derivative of the function at $x = 1$.

Solution: $f'(x) = 2x - 2$ if $x < 1$ and $f'(x) = 2 - 2x$ if $x \geq 1$. For both of these, if we substitute $x = 1$, we get a value of 0. This implies that there is a horizontal tangent line for the function at $x = 1$. But does this make sense? Again, examine the graph of the function.

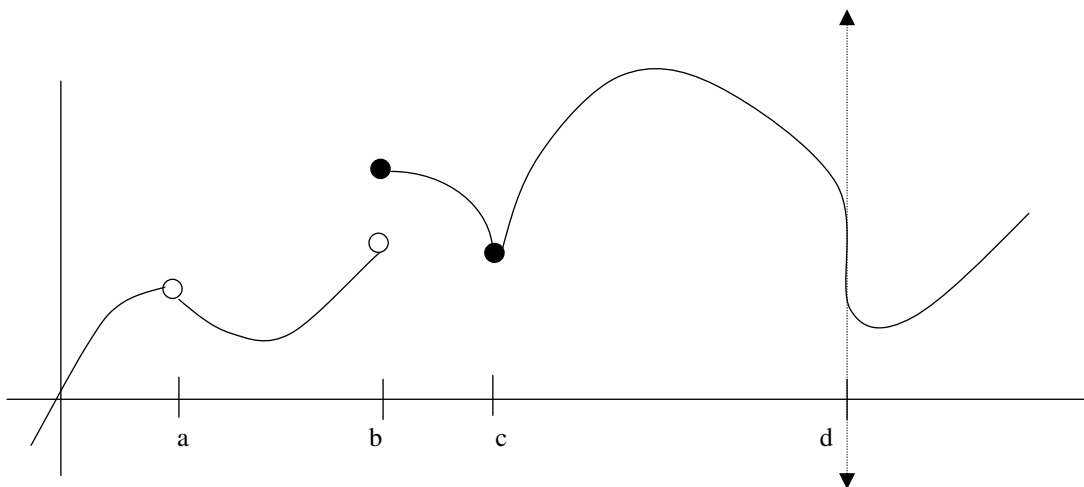


There is a discontinuity at $x = 1$. It doesn't make sense that there is a tangent line at a point of discontinuity.

These three examples illustrate three ways that differentiability can be destroyed:

1. When the graph of a function comes to a sharp point. There is no tangent line at this point.
2. When the derivative of a function is undefined at a particular value and the graph of the function becomes so steep that it appears almost vertical itself. The tangent line is a vertical line with an undefined slope.
3. When the graph of a function is not continuous at a value. There is no tangent line at this point.

These situations are illustrated below.



At points a and b, the function is discontinuous and therefore not differentiable. There are no tangent lines to the curve at these points. At point c, the function is continuous but not differentiable since the curve comes to a sharp point. There is no tangent line at this point. At point d, the function is again continuous, but not differentiable. However, there is a vertical tangent line at this point.

Homework Exercises Part 5: Differentiability

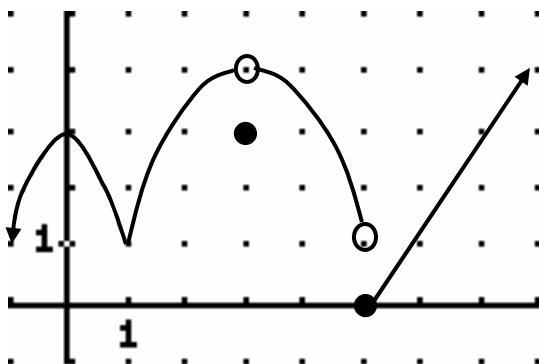
Examine the graphs of each of the following functions. Indicate the points at which the functions are not differentiable, and give the reason this occurs.

- | | |
|---------------------------|---------------------------------|
| 1. $f(x) = 1 - x $ | 5. $f(x) = \frac{2x}{x-1}$ |
| 2. $f(x) = \sqrt[5]{x^4}$ | 6. $f(x) = \sqrt[3]{(x-3)^2}$ |
| 3. $f(x) = \frac{1}{x}$ | 7. $f(x) = \sqrt[4]{x}$ |
| 4. $f(x) = x^2 - 4 $ | 8. $f(x) = \frac{x^2}{x^2 - 4}$ |

9. Determine whether the functions below are differentiable at $x = 2$.

a. $f(x) = \begin{cases} x^2 + 1 & \text{if } x \leq 2 \\ 4x - 3 & \text{if } x > 2 \end{cases}$	b. $g(x) = \begin{cases} \frac{1}{2}x + 1 & \text{if } x < 2 \\ \sqrt{2}x & \text{if } x \geq 2 \end{cases}$
--	--

10. Consider the pictured function below:



- Where on the interval $-1 < x < 7$ does the limit of the function fail to exist?
- Where on the interval $-1 < x < 7$ does the function fail to be continuous?
- Where on the interval $-1 < x < 7$ does the function fail to be differentiable?
- Where on the interval $-1 < x < 7$ is $f'(x) = 0$?

Determine if the following statements are true or false.

- If $f'(x) = g'(x)$ for all x , then $f(x) = g(x)$ for all x .
- If $f(x) = \pi^4$, then $f'(x) = 4\pi^3$.
- If $f'(c)$ exists, then f is continuous at c .
- The graph of $y = \sqrt[3]{x}$ has a tangent line at $x = 0$ but the derivative of the function does not exist there.
- The derivative of a polynomial is a polynomial.
- If a function is continuous at a point, then it is differentiable at that point.
- If a function has derivatives from both the right- and the left-sides of a point, then it is differentiable at that point.

Part 6: Other Notations and Higher Order Derivatives

The $f'(x)$ notation is one of the most common notations for derivatives, but there are others:

$$\frac{dy}{dx}, \quad D_x f, \quad \frac{d}{dx}(f)$$

These three notations use “ d ” or “ D ” to indicate taking the derivative of a function named y or f : the derivative of y with respect to x or the derivative of f with respect to x . In these cases, y or f is the dependent variable; x is the independent variable.

Example 1: Find each of the indicated derivatives: a. $\frac{d}{dt}(2t^3 - \sin t)$ b. $D_x(ax^2 + bx + c)$

Solutions: For part a, you are being asked to find the derivative of the given function with respect to the independent variable t : $\frac{d}{dt}(2t^3 - \sin t) = 6t^2 - \cos t$

For part b, you are being asked to find the derivative of the given function with respect to the independent variable x . This indicates that a , b , and c are constants.

$$D_x(ax^2 + bx + c) = 2ax + b$$

There are several applications for higher order derivatives; i.e., derivatives of derivatives. We will examine these applications in the next chapter. It is important to learn the notation and practice the process before learning the applications. The derivative of the first derivative is the second derivative. It can be denoted by

$f''(x)$, $D_x^2 y$ or $\frac{d^2 y}{dx^2}$. A summary of notations is given below.

Derivative	f notation	D notation	$\frac{dy}{dx}$ notation
first	$f'(x)$	$D_x y$	$\frac{dy}{dx}$
second	$f''(x)$	$D_x^2 y$	$\frac{d^2 y}{dx^2}$
third	$f'''(x)$	$D_x^3 y$	$\frac{d^3 y}{dx^3}$
fourth	$f^{iv}(x)$	$D_x^4 y$	$\frac{d^4 y}{dx^4}$
fifth	$f^v(x)$	$D_x^5 y$	$\frac{d^5 y}{dx^5}$
n^{th}	$f^{(n)}(x)$	$D_x^n y$	$\frac{d^n y}{dx^n}$

Example 2: Find the first five derivatives of $f(x) = \sin x$

Solution:

$$f'(x) = \cos x$$

$$f''(x) = -\sin x$$

$$f'''(x) = -\cos x$$

$$f^{iv}(x) = \sin x$$

$$f^v(x) = \cos x$$

Example 3: Find the first four derivatives of $y = \ln x$

Solution:

$$\frac{dy}{dx} = \frac{1}{x}$$

$$\frac{d^2 y}{dx^2} = -\frac{1}{x^2}$$

$$\frac{d^3 y}{dx^3} = \frac{2}{x^3}$$

$$\frac{d^4 y}{dx^4} = -\frac{6}{x^4}$$

One type of problem that we have studied throughout the last two units has been position and velocity. We learned that the change in position (x) with respect to time (t) yielded velocity; i.e., velocity = $\frac{dx}{dt}$. The change in velocity

(v) with respect to time yields acceleration; i.e., acceleration = $\frac{dv}{dt} = \frac{d^2x}{dt^2}$

Example 4: An object moves along a path so that its position x is defined by the function $x(t) = 2t^3 - 24t + 8$, where x is measured in centimeters and t in seconds ($t > 0$). Determine the velocity when $t = 1$ and $t = 6$. When is the velocity 0? Determine the acceleration at $t = 1$.

Solution: Velocity = $\frac{dx}{dt} = 6t^2 - 24$. So, at $t = 1$, the velocity is -18 cm/sec. At $t = 6$, the velocity is 192 cm/sec. The velocity is 0 when $6t^2 - 24 = 0$. Solving this equation, we get $6t^2 - 24 = 6(t^2 - 4) = 6(t - 2)(t + 2) \Rightarrow t = \pm 2$. Since $t > 0$, the velocity is 0 at $t = 2$ seconds. Acceleration = $\frac{dv}{dt} = 12t$. At $t = 1$, the acceleration is 12 cm/sec².

Homework Exercises Part 6: Other Notations and Higher Order Derivatives

Find the first three derivatives for each of the following functions.

1. $y = x^3 - 3x^2 + 6x - 5$

5. $f(x) = \sqrt{x}$

2. $y = \cos x$

6. $f(x) = \frac{1}{\sqrt{x}}$

3. $y = e^x$

7. $y = \ln(x^2)$

4. $f(x) = \frac{1}{x^2}$

8. $f(x) = \sin x + \cos x$

9. Without using any formulas, find the following derivatives. Explain your reasoning.

a. $D_x^5(x^4 - 3x^2 + 5x - 7)$

b. $D_x^5(\sin x)$

c. $D_x^9(e^x)$

10. Suppose $f(x) = ax^2 + bx + c$ and $f(1) = 5$, $f'(1) = 3$, $f''(1) = -4$. Find a , b , and c .

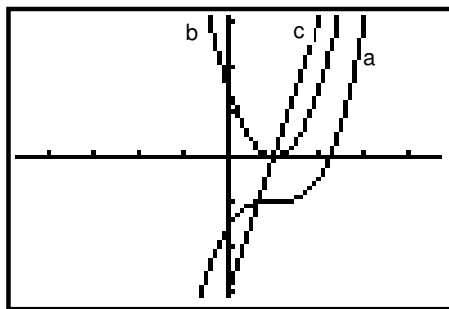
11. The position of an object moving along a coordinate line is given by $x(t) = 2 - 2 \sin t$. Find the velocity and acceleration of the object at time $t = \frac{\pi}{4}$.

12. The position of an object thrown upward on the moon is given by the function $x(t) = -2.7t^2 + 27t + 6$ where x is measured in feet and t is measured in seconds.

a. Find expressions for the velocity and acceleration of the object.

b. Find the time when the object is at its highest point by finding the time when the velocity is 0. What is the height at this time?

13. The graphs of $f(x)$, $f'(x)$ and $f''(x)$ are shown. Choose the letter (a, b, or c) that represents each.



True/False

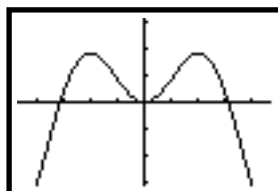
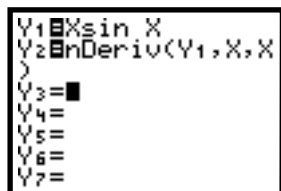
14. If $y = (x-1)(x+2)(x+5)(x-7)$ then $\frac{d^5 y}{dx^5} = 0$.

15. The second derivative represents the rate of change of the first derivative.

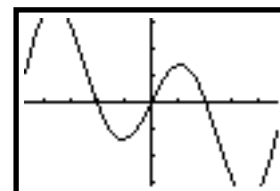
16. If the velocity of an object is constant, then its acceleration is 0.

Part 7: Other Differentiation Rules

Consider the function, $y = x \sin x$. This is a product of two functions for which we know the derivatives. We also know that the derivative of the sum of two functions is the sum of the derivatives. Is this true of the product of two derivatives? If it was true that the derivative of the product of two functions is the product of the derivatives of the two functions, then the derivative of $x \sin x$ should be $(1)(\cos x)$. We can check this graphically with the calculator. The function and its derivative are pictured below.



Y1

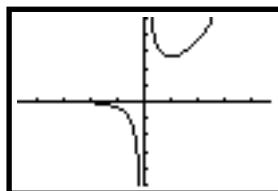
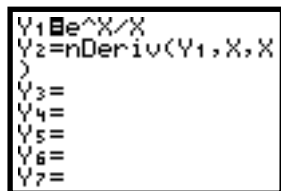


Y2

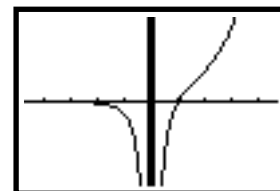
It is obvious that the graph of the derivative of $x \sin x$ is not $\cos x$. Therefore, the derivative of the product of two functions is NOT the product of the derivatives of the two functions.

What of the derivative of the quotient of two functions? Consider the function $y = \frac{e^x}{x}$. If the derivative of the quotient of two functions is the quotient of the derivatives of the two functions, then the derivative should be or

$$\frac{e^x}{1} = e^x$$



Y1



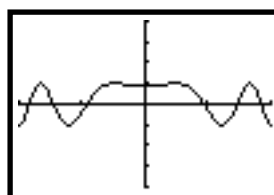
Y2

This is not the graph of e^x . Therefore, the derivative of the quotient of two functions is NOT the quotient of the derivatives of the two functions.

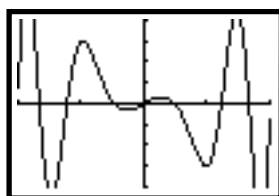
What of the derivative of the function of a function (the composite of two functions)? Consider the function $y = \sin(x^2 + 1)$. Is the derivative of this function simply $\cos(x^2 + 1)$?

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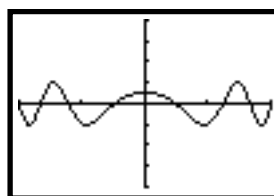
Y1=sin(X^2+1)
Y2=Deriv(Y1,X,X)
)
Y3=
Y4=
Y5=
Y6=
Y7=
    
```



Y1



Y2



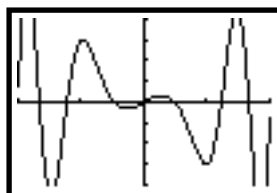
$y = \cos(x^2 + 1)$

Again, the rule is not so simple. The derivative of the composite of two functions is NOT simply the derivative of the principal function composed with the given function. A special rule called the “chain rule” is needed in this case.

Chain Rule: If f is differentiable at x , g is differentiable at $f(x)$, and $h(x) = g(f(x))$, then h is differentiable at x and $h'(x) = g'(f(x)) \cdot f'(x)$.

For the example above, if we let $f(x) = x^2 + 1$ and $g(x) = \sin x$, then $h(x) = g(f(x))$, $f'(x) = 2x$, and $g'(x) = \cos x$. Therefore, according to the rule, $h'(x) = \cos(f(x)) \cdot 2x = \cos(x^2 + 1) \cdot 2x = 2x \cos(x^2 + 1)$.

The graph of this function is pictured below.



Note that it is the same function pictured in Y2 above, the derivative of $y = \sin(x^2 + 1)$

So, in order to apply the Chain Rule, a composite function must be decomposed into its component parts, the derivatives of each of these component parts must be determined and the Rule applied.

Example 1: Find the derivative of $h(x) = \sqrt{4-x^2}$

Solution: Let $f(x) = 4-x^2$ and $g(x) = \sqrt{x} = x^{\frac{1}{2}}$.

Thus, $f'(x) = -2x$ and $g'(x) = \frac{1}{2}x^{-\frac{1}{2}}$.

Using the rule, we get

$$h'(x) = \frac{1}{2}(f(x))^{-\frac{1}{2}} \cdot (-2x) = \frac{1}{2}(4-x^2)^{-\frac{1}{2}} \cdot (-2x) = \frac{-x}{\sqrt{4-x^2}}$$

Example 2: Find the derivative of $h(x) = e^{2x}$

Solution: Let $f(x) = 2x$ and $g(x) = e^x$. Thus, $f'(x) = 2$ and $g'(x) = e^x$.

Using the rule, we get $h'(x) = e^{f(x)} \cdot (2) = e^{2x} \cdot (2) = 2e^{2x}$

The two rules yet to be learned are the product and quotient rules; and, the chain rule must be more completely explored. In combination with the rules already learned, they will allow you to find the derivative of any function. The product and quotient rules will be studied next year in calculus, as well as combinations of all the rules in much more complicated functions.

Homework Exercises Part 7: Other Differentiation Rules

For problems 1 - 6, find $h'(x)$.

1. $h(x) = (2x+1)^{10}$

4. $h(x) = (\cos x)^2$

2. $h(x) = \ln(x^3 - 1)$

5. $h(x) = \sin(x^2)$

3. $h(x) = e^{\sin x}$

6. $h(x) = \frac{1}{\sqrt{x^2 + 4}}$

For problems 7 - 16, determine if the derivatives rules from this chapter apply. If they do, find the derivative. If they do not, indicate why.

7. $y = \sqrt{3x-1}$

12. $y = \sin^2 x$

8. $y = \frac{1}{x^2} - \pi$

13. $y = \frac{1}{2x^3 - x}$

9. $y = 2^x$

14. $y = e^4 x$

10. $y = x^x$

15. $y = (\ln 2)e^x$

11. $y = ex^4$

16. $y = \pi^x + e^x$

True/False

17. If $y = \frac{1}{3x}$ then $\frac{dy}{dx} = \frac{-3}{x^2}$.

18. If $y = \pi^2$, then $\frac{dy}{dx} = 2\pi$.

19. If $y = \frac{x}{\pi}$, then $\frac{dy}{dx} = \frac{1}{\pi}$.

20. If $f(x) = \sqrt{2x}$, then $f'(x) = \frac{1}{\sqrt{2x}}$.

21. If $y = \sqrt{1+x}$, then $\frac{dy}{dx} = \frac{1}{2\sqrt{1+x}}$.

Optional Exercises:

22. Using the function $f(x) = x \sin x$, verify the product rule: $(fg)' = f'g + fg'$.

23. Using the function $f(x) = \frac{e^x}{x}$, verify the quotient rule: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

Homework Exercises Review

Find the derivative of each of the following:

1. $f(x) = 4x^3 - 3x^2$

2. $f(x) = \frac{2x^4 - 1}{x^2}$

3. $f(x) = \sqrt{x} - \frac{1}{\sqrt{x}}$

4. $f(x) = \frac{2}{3x^2}$

5. $f(x) = (x-1)(x^2 - 2x + 3)$

6. $f(x) = \ln\left[(2x^3)(2e^x)\right]$

7. $f(x) = \sin x - 2 \cos x - \frac{1}{x}$

8. $f(x) = 3e^x - 2 \ln x$

9. $f(x) = \frac{\pi}{x^3} - \pi^2$

10. $f(x) = (\tan x)(\cos x)$

11. $f(x) = \frac{\sqrt{x}}{x^2}$

12. $f(x) = \frac{x^2 + 1}{\sqrt{x}}$

Find the second derivative for each of the following:

13. $f(x) = x^2 - \frac{2}{x}$

14. $f(x) = (2x-1)(3x+4)$

15. $f(x) = 2 \sin x - 3 \cos x$

16. $f(x) = \ln(x^4)$

17. Find the equation of the normal line and the tangent line to the curve $y = \ln x$ at $x = 2$.

18. Sketch the graph of $f(x) = 4 - |x - 2|$.

- Is f continuous at $x = 2$? Why or why not?
- Is f differentiable at $x = 2$? Why or why not?

19. Sketch the graph of $f(x) = \begin{cases} x^2 + 4x + 2 & \text{if } x < -2 \\ 1 - 4x - x^2 & \text{if } x \geq -2 \end{cases}$

- Is f continuous at $x = -2$? Why or why not?
- Is f differentiable at $x = -2$? Why or why not?

20. Let f be the real-valued function defined by $f(x) = \sqrt{x}$

- Give the domain and range of f .
- Determine the slope of the line tangent to the graph of f at $x = 4$.
- Determine the y -intercept of the line tangent to the graph of f at $x = 4$.
- Give the coordinates of the point on the graph of f where the tangent line is parallel to $y = x - 2$.

21. AP Calculus Problem: 1978 AB 1 Given $f(x) = x^3 - x^2 - 4x + 4$. The point (a, b) is on $f(x)$ and a tangent line passes through (a, b) and $(0, -8)$, which is not on the graph of $f(x)$. Find a and b .

True/False

- If a function is continuous, then it is differentiable.
- If $f(x) = g(x) + c$, then $f'(x) = g'(x)$.
- If $f(x)$ is an n^{th} degree polynomial, then $f^{(n)}(x) = 0$.
- If $f(x)$ is an n^{th} degree polynomial, then $f^{(n+1)}(x) = 0$.
- The acceleration of an object can be negative.
- If $f(x) = \frac{1}{x}$, then $f^{(4)}(x) = \frac{24}{x^5}$.
- If $f(x)$ is differentiable at $x = c$, then $f'(x)$ is differentiable at $x = c$.
- If a graph of a function has a tangent line at a point, then it is differentiable at that point.
- If $y = f(x) \cdot g(x)$, then $\frac{dy}{dx} = f'(x) \cdot g'(x)$.