Part 1: Vocabulary of Vectors

1. Draw a vector beginning at the point A(-3, 2) and going to point B(1, -1).

   a. On your diagram, label the initial point and the terminal point.

   b. Find the magnitude of the vector. Show work.

   c. Write the vector in component form.

   d. Find the direction of the vector. Show work.

   e. Draw the vector $\overrightarrow{AB}$ in standard position on the grid at the left.

   f. Draw a vector opposite to this vector.

   g. Write the component form of the opposite vector.

   h. What is the direction of the opposite vector?

2. Given vectors **v**(2,3) and **w**(-4, -2).

   a. Draw the vector on the grid at the right.

   b. Draw the sum, **v + w**.

   c. Find the sum (resultant) of the two vectors using components. Express your answer in the form a**i** + b**j**.

   d. Find the angle between the two vectors. (Hint: Use the Law of Cosines). Show all work.

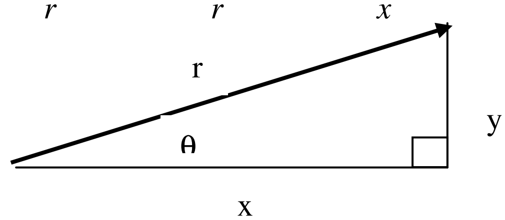
Part 2: Vector Components

Any vector can be written as a linear combination of two nonparallel vectors. We know that the vector 4**i** - 3**j** has two components: a horizontal component of length 4 in the positive x direction and a vertical component of length 3 in the negative y direction. Consider the problems you have solved geometrically using the Laws of Sines and Cosines. How could components be used to solve these types of force or navigation problems? Consider the following example.

Example 1: A force F of 18 N (newtons) is applied at an angle of 37° with the horizontal. Resolve F into its horizontal and vertical components.

Solution: The solution is based on the right triangle trigonometry learned in Geometry, SOHCAHTOA. If the hypotenuse is r, the horizontal is x and the vertical is y then the following relationships are true:

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$



Therefore,

$$y = r\sin\theta \qquad x = r\cos\theta \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

For this particular problem, r = 18 and θ = 37°. So, x = 18(cos 37°) and y = 18(sin 37°).
Rounded to three decimal places the values are x = 14.375 and y = 10.833.
The vector F given magnitude 18 and angle of 37° measured from the positive x-axis has a horizontal component of 14.375**i** and a vertical component of 10.833**j**; i.e., F = 14.375**i** + 10.833**j**. Therefore, a vector can be expressed in the polar system in terms of its magnitude and direction, (r, θ) or in the Cartesian system in terms of its **i** and **j** components, (x, y).

Exercises:

1. Resolve each vector, (r, θ), with given magnitude and direction from the horizontal into its components, (x, y).

   a. (13.5, 80°) b. (25, 241°)

2. Given the horizontal and vertical components (x, y), write the vector in terms of its magnitude and direction, (r, θ).

   a. (-7, 4) b. (3, -9)

3. A downstream current and a cross-stream wind current at 4 km/h give a sailboat an effective speed of 12 km/h. What is the speed of the downstream current and the angle between the wind current and the path of the sailboat?

Part 3: Vectors in the Plane and Graphing Calculators

Navigation problems are easily solved using vectors. There are three terms commonly used in navigation problems: bearing is the angle measured from due North, clockwise; direction is measured from the positive x-axis, counterclockwise; and, heading must state one of the following descriptions

N of W	*S of W*	*E of N*	*W of N*
S of E	*E of S*	*W of S*	*N of E*

A typical vector problem is given below. The steps you should use to solve the problem are outlined.

Class Example: A plane is flying on a bearing of 30° at 300 mph. A tail wind is blowing in the direction of 40° south of east at 60 mph. Determine the actual speed and direction of the plane.

Step 1: Draw a diagram illustrating the problem.

Step 2: Write each vector in component form:

The horizontal component for the plane is $x = r\cos\theta = 300\cos 60^{\circ} = 150$

The vertical component for the plane is $y = r\sin\theta = 300\sin 60^{\circ} \approx 259.808$

The horizontal component for the wind is $x = 60\cos(-40^{\circ}) \approx 45.963.$

The vertical component for the wind is $y = 60\sin(-40^{\circ}) \approx -38.567$

Step 3: Add the components to find the resultant: <195.963, 221.241>; i.e., the resultant vector can be expressed as 195.963**i** + 221.241**j**.

Step 4: Convert the vector to polar form in order to answer the question.

The second and third steps can be done your calculator using the parametric (degree) mode. Follow the steps below:

X1T = (300cos60)T   X2T = (60cos(-40))T
Y1T = (300sin60)T   Y2T = (60sin(-40))T

The resultant is X3T = X1T + X2T
Y3T = Y1T + Y2T

Set the window for this problem to be T $\in$ [0,1,1], X $\in$ [0, 300, 100] and Y $\in$ [-100, 300, 25].
Trace X3T to T = 1. Note the results are <195.963, 221.241>. You can complete the problem as above or go into your calculator **WINDOW** **FORMAT** and change **RectGC** to **PolarGC**.
Now when you trace out to T = 1, you will get your final answer.

So, the solution to this problem is a vector with magnitude 295.548 mph and direction 48.467°.

Another common vector application involves forces. A force vector is a vector that represents the direction and magnitude of an applied force.

Example: A man and a horse are both pulling on a large rock. The man is pulling with a force of 200 pounds and the horse is pulling with a force of 1000 pounds 50¡ away from the direction of the pull of the man. Find the resultant force by using the steps outlined above.

Step 1: Diagram

Step 2. Write vectors:

Horse: x = ___________________ y = ___________________

Man: x = ___________________ y = ___________________

Step 3. Add the components: ______________________________________

Step 4. Convert to polar form: ____________________________________

Answer the question: ____

Exercises: Solve each of the following problems. For problems 1-4, use the four steps outlined above. Show all work.

1. A ship steams 100 miles east, and then 40 miles on a bearing of 120°. How far has the ship traveled and what is its final bearing?

2. A ship steams 45 miles due north, turns to a bearing of 45° and steams 25 miles, then turns south and steams 40 miles. How far is it from its starting place, and what is its bearing from its starting place?

3. Two forces, one of 100 pounds and the other of 150 pounds, act on the same objects, at angles of 20° and 60°, respectively, with the positive x-axis. Find the direction and magnitude of the resultant vector of these two forces.

4. Three forces of 75 pounds, 100 pounds, and 125 pounds act on the same object at angles of 30°, 45°, and 120°, respectively, with the positive x-axis. Find the direction and magnitude of the resultant of these forces

5. Find the measure of the angle between $u = -2\mathbf{i} - 9\mathbf{j}$ and $v = \mathbf{i} + 7\mathbf{j}$.

6. Determine if the following vectors are parallel or perpendicular or neither. If neither, give the measure of the angle between them.

a. $u = -2\mathbf{i} - 9\mathbf{j}$ and $v = 9\mathbf{i} + 2\mathbf{j}$

b. $u = -2\mathbf{i} - 9\mathbf{j}$ and $v = 4\mathbf{i} + 18\mathbf{j}$

c. $u = -2\mathbf{i} - 9\mathbf{j}$ and $v = 1\mathbf{i} + 9\mathbf{j}$

d. $u = -2\mathbf{i} - 9\mathbf{j}$ and $v = 9\mathbf{i} - 2\mathbf{j}$

Part 4: Vectors in Space

The directed line segments that we use to represent forces, displacements and velocities in space are called vectors, just as they are in the plane. The same rules of addition subtraction, and scalar multiplication apply.

The vectors from the origin to the points (1, 0, 0,) , (0, 1, 0) and (0, 0,1) are the basic unit vectors. We denote them by **i**, **j**, and **k**. The position vector r from the origin O to the typical point P(x, y, z) is **r** = $\overrightarrow{OP}$ = x**i** + y**j** + z**k**

The values x, y, z are the scalar components of the position vector.

Class Notes and Examples:

1. Using what you know of vectors in the plane, extend this knowledge to draw a vector in three space with initial point at the origin and terminal point at

a. (4, 7, 5)

b. (3, -2, 6)

2. Find the magnitude of each of the above vectors by extending the distance formula from two-dimensions to three-dimensions.

a.

b.

3. For each pair of points A and B, find an ordered pair that represents $\overrightarrow{AB}$. Then find the magnitude of each vector.

a. A(-4, 5, 8), B(7, 2, -9)

b. A(6, 8, -5), B(7, -3, 12)

4. Write each of the following vectors as a linear combination using basic unit vectors.

a. <8, -9, 2>

b. <7, 6, -5>

5. Given vectors v = 2i - 4j + 5k and w = 6i - 8j + 9k, find

a. v + w

b. v - w

c. 4v + 3w

d. 5v - 2w

6. Given a vector v = 3i + 4j - 5k,

a. Find |v|.

7. In three dimensions, a unit vector going in the same direction as v is called its direction. Find the direction and magnitude of a vector extending from A(2, 1, 3) to B(-4, 5, 7).

Exercises:

Find the lengths and directions of the vectors in Exercises 1 - 6

1. 3i + j - 2k
2. i + 4j - 9k
3. 6k
4. - 3j
5. (-1/3)j + (1/2)k
6. (1/Ã5)i - (1/Ã5)j - (1/Ã5)k

In Exercises 7 - 9, find the direction of the vector from P1 to P2.

7. P1(1, -1, 2), P2(3, 4, 0)
8. P1(1, - 3, 5), P2(4, -2, 8)
9. P1(0, 0, 0), P2(5, -2, -2)

10. Find the vectors whose lengths and directions are given. Sketch the vector.

	Length (|v|)	Direction (v/ |v|)
a.	3	j
b.	Ã2	- i
c.	1/3	(4/5)j - (3/5)k
d.	17	(6/17)i - (2/17)j + (3/17)k

11. Find a vector of magnitude 7 in the direction of v = 5i - 12k.

12. Find a vector 5 units long in the direction opposite to the direction of v = 2i - 3j + 6k.

Part 5: Dot Products

We now introduce the dot product of two vectors, the first of two products in the study of vectors. Our motivation is the need to calculate the work done by a constant force in displacing a mass. When the force and displacement are represented as vectors, the dot product of the two vectors gives the work done by the force during the displacement.

Dot products, also called scalar products because the resulting products are numbers and not vectors, have applications in mathematics as well as in engineering and physics. In this section, we present the algebraic and geometric properties on which many of these applications depend. The second kind of vector product, the cross product, will be described in the next section.

DEFINITION: The dot product v . w (Òv dot wÓ) or scalar product of two vectors v and w is the number v . w = |v| |w| cos q, where q measures the angle (0 ² q ² ¹) made by representative v and w vectors with the same endpoint (as in the diagram below).

This gives us a handy way to calculate a vectorÕs length, as we will see.

Calculation

To calculate v . w from components of v and w, we let v = a1i + a2j + a3k and w = b1i + b2j + b3k and z = w - v = (b1 - a1)i + (b2 - a2)j + (b3 - a3)k. Then we apply the law of cosines to a triangle (see the figure below) whose sides represent the vectors v, w, and z and obtain

ALTERNATIVE DEFINITION: The dot product v . w (Òv dot wÓ) or scalar product of two vectors v and w is the number

v . w = a1b1 + a2b2 + a3b3

Thus, to find the dot product of two given vectors, we multiply their corresponding i, j, and k components and add the results. When we manipulate the two formulas, it is possible to get a formula for finding the angle between two nonzero vectors.

Example 1: Find the angle between v = 2i + j + 2k and w = 3i - 6j - 2k.

Laws of Multiplication

From the equation v . w = a1b1 + a2b2 + a3b3 (the alternative definition of the dot product), we can see immediately that vector multiplication is commutative:

COMMUTATIVE RULE: v . w = w . v

Dot products also obey a distributive law.

DISTRIBUTIVE RULE: v . (w + z) = v . w + v . z

Orthogonal Vectors

Two vectors whose dot product is zero are said to be orthogonal or perpendicular.

DEFINITION: Vectors v and w are orthogonal if v . w = 0.

The zero vector 0 = 0i + 0j + 0k is orthogonal to every vector because its dot product with every vector is zero. When neither |v| nor |w| is zero, the equation v . w = |v| |w| cos q tells us that v . w is zero if and only if cos q is zero, that is when q equals ¹/2.

Example 2: The vectors v = 4i + 5j - 2k and w = 3i - 2j +4k are orthogonal because

v . w = (4)(3) + (5)(- 2) + (- 2)(1) = 0

Work

In mechanics, the work done by a constant force F when the point of application undergoes a displacement from A to B (see the figure below) is defined to be the dot product of F with the vector from A to B.

Example 3: If |F| = 30 newtons (about 8 pounds), |AB| = 4 m, and q = 50¡, the work done by F in acting from A to B is

Work = |F| |AB| cos q
= (30)(4) cos 50¡
~ 77.135 newton-meters.

Exercises: In Exercises 1 - 6, find v . w, |v|, |w|, and the angle between v and w.

1. v = 4i + 2j, w = 5j + k
2. v = 3i - 4j - k, w = -2j
3. v = 5j - 2k, w = i + j - k
4. v = - i + j, w = Ã3 i + Ã5 j + 2k
5. v = 3i - 4j + Ã7 k, w = -2i + 4j - Ã7 k
6. v = 12i + 7j - 2k, w = 3j

7. Cancellation in dot products is risky. In real-number multiplication, if ab1 = ab2 and a is not zero, we can safely cancel the a and conclude that b1 = b2. Not so for dot product multiplication -- if v . w1 = v . w2 and v - 0, it is not safe to conclude that w1 = w2. See whether you can come up with an example. Keep it simple: Experiment with i, j, and k.

8. Find the work done by a force F = -7k (magnitude 7 newtons) in moving an object along the line from the point (1, 2, 1) to the origin (distance in meters).

9. A locomotive exerts a constant force of 50,000 newtons (N) on a freight train while pulling it 3 km along a straight track. How much work does the locomotive do?

10. How much work does it take to slide a crate 15 m along a loading dock by pulling on the crate with a 175 - N force at an angle of 38¡ from the horizontal?

11. The wind passing over a boatÕs sail exerts a 5000 - N force F as shown below. If the force vector makes a 65¡ angle with the line of the boatÕs forward motion, how much work does the wind perform in moving the boat forward 2 km?

Part 6: Cross Products

The second of the two products that we will learn is the cross product or vector product. Our motivation is the need to calculate such quantities as torque. When we turn a bolt by applying a force to a wrench, the torque is the force that we produce acting along the axis of the bolt to drive the bolt forward. Vector products are also widely used to describe the effects of forces in studies of electricity, magnetism, fluid mechanics, and planetary motion.

Mathematically, the cross product of two vectors v and w produces a vector perpendicular to the plane containing v and w.

The formula can be remembered as the terms in the expansion of the determinant

It is also possible to write a program to calculate the cross product of two vectors.

Example 1: Find v x w and w x v if v = 3i - 2j + k and w = 4i + j + k.

Written in determinant form,

$$= -3i + j + 11k$$

Calculating w x v in a similar manner yields the result 3i - j - 11k.

Unlike the dot product, the cross product is not a commutative operation. Reversing the order of the factors in a nonzero vector product reverses the direction of the resulting vector. Thus,

If one of both of v and w is zero, we define v x w to be zero as well. Thus, the cross product of two vectors v and w is zero if and only if v and w are parallel or one or both of them is the zero vector.

The Associative and Distributive Laws

As a rule, cross product multiplication is not associative, but the scalar distributive law

(rv) x (sw) = (rs)(v x w)

does hold, as do the vector distributive laws

v x (w + z) = (v x w) + (v x z) and
(w + z) x v = (w x v) + (z x v)

|v x w| is the Area of a Parallelogram

The area of the parallelogram having v and w as adjacent sides is |v x w|. See the figure below.

Example 2: Find the area of the parallelogram formed by v = 0i + 3j + k and B = -3i + 5j - 2k.

= -11i - 3j + 9k

Example 3: Find the area of the triangle with vertices C(1, -1, 0), A(2, 1, -1), and B(-1, 1, 2).

Note that these are the same vectors from Example 2, so the triangleÕs area is half the area of the parallelogram or 3Ã2 ~ 4.243.

The Test for Parallelism

Example 4: Determine if the following pairs of vectors are parallel

(a) v = i + 2j - k and w = -2i - 4j + 2k
(b) v = i + 2j - k and w = -2i + 2j + 2k

= (4 - 4) i - (2 - 2)j + (-4 + 4)k = 0i - 0j + 0k = 0. They are parallel.

(b) See Example 2. v x w = 6i + 6k - 0. They are not parallel.

(c) Run the program from Appendix 2.

The Triple Scalar or Box Product

The product (v x w) . z is called the triple scalar product of v, w and z (in that order). The product is the volume of the parallelepiped (parallelogram-sided box) determined by v, w and z (see the figure below). The number |v xw| is the area of the base of the parallelogram, and |z|cosq is the height of the box. If q is greater than 90¡, then cos q is negative, and we must take the absolute value of
(v x w) . z to get the volume. Because of the geometry of this situation, (v x w) . z is often called the box product of v, w and z.

Example 5: Find the volume of the parallelepiped determined by v = i + 2j - k, w = -2i + 3k, and z = 7j - 4k.

The volume is the absolute value of

So the volume is 23.

Exercises:

Find the cross product, v x w

1. v = 2i - 2j - k, w = i - k
2. v = 2i - 2j + 4k, w = -i + j - 2k
3. v = 2i, w = -3j
4. v = -8i - 2j - 4k, w = 2i + 2j + k

Find the area of triangle ABC

5. A(1, -1, 2), B(2, 0, -1), C(0, 2, 1)
6. A(2, -2, 1), B(3, -1, 2), C(3, -1, 1)

7. Let v = 5i - j + k, w = j - 5k, z = -15i + 3j - 3k. Which vectors if any, are
(a) perpendicular, (b) parallel?

8. Let v = i + 2j - k, w = -i + j + k, z = i + k, D = -3i - 6j + 3k. Which vectors, if any, are
(a) perpendicular, (b) parallel?

9. If v = 2i - j and w = i + 3j - 2k, find v x w. Then calculate (v x w) . v and (v x w) . w.

10. Is (v x w) . v always zero? Explain. What about (v x w) . w?

11. Find the volume of the parallelepiped determined by v, w, and z.

a. v = 2i, w = 2j, z = 2k

b. v = 2i + j, w = 2i - j + k, z = i + 2k

Review Exercises:

1. Let A = (2, -1), B = (3, 1) and C = (- 4, 2)

2. (from the textbook, p 473). Let u = <2, -1>, v = <4, 2> and w = <1 -3> be vectors and determine each of the following:

a. u - v
b. |u + v|
c. 2u - 3w
d. |w - 2u|

3. (from the textbook, p 469). A plane is flying on a bearing 10¡ east of south at 460 mph. A tail wind is blowing in the direction 20¡ west of south at 80 mph. Express the actual velocity of the plane as a vector. Determine the actual speed and direction of the plane.

4. Determine if the vectors are perpendicular and show your method: u = (3, 5) and v = (- 5, 3).

5. If u = 2i - 3j + k and v = 3i - j + 5k, find

a. the length and direction of u
b. a vector of magnitude 3 in the direction of v
c. the dot product of u and v. Are these vectors perpendicular? Explain.

6. Vectors A and B are defined as A = 3i + 4j - k and B = 10i - 3j + 4k. Find

a. A . B
b. |A|
c. |B|
d. the angle between A and B.

7. How much work does it take to push a carton 100 m using a 10 - N force at an angle of 30¡ from the horizontal?

8. Given A = 3i + 2j + 4k and B = - i - 3j + 2k, find

a. the cross product of A and B. Tell what method you used.
b. Find the area of the triangle formed by these two vectors.

9. Find the volume of the parallelepiped determined by A = 2i + 3j, B = 5j + 3k and C = i + 2k.

Appendix 1

The program below calculates the cross product of two vectors.

```
PROGRAM:CRSSPROD
:ClrHome
:Disp ÒFIRST VECTOR, A1=Ó
:INPUT A
:Disp ÒA2=Ó
:INPUT B
:Disp ÒA3=Ó
:INPUT C
:Disp ÒSECOND VECTOR, B1=Ó
:INPUT D
:Disp ÒB2=Ó
:INPUT E
:Disp ÒB3=Ó
:INPUT F
:BF-EC® I
:DC-AF® J
:AE-BD® K
:Disp ÒCROSS PRODUCTÓ
:Disp ÒCOEFF  OF  I=Ó
:Disp I
:Disp ÒCOEFF  OF  J=Ó
:Disp J
:Disp ÒCOEFF  OF  K=Ó
:Disp K
```