17 Basic Counting Arguments

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May 12, 2006

The first 3 arguments are incredibly basic, and should be familiar to everyone already. The next 8 arguments are fundamental and should be learned by all who wish to solve combinatorics problems. The next 4 are more advanced topics for those who have had prior experience in combinatorics; these will probably not be required for an ARML problem, but may make such a problem much more simple. The last 2 problems are highly advanced arguments at the olympiad level that will probably not appear on an ARML problem, but that have been included because the authors wish to put their solutions in the solutions section.

1. How many ways can one arrange the elements of a set of size n?

   $\boxed{n!}$

   You can place the first element in any of n places. You can place the second element in any of $n-1$ places. You can place the kth element in any of $n-k+1$ places. Thus there are a total of $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1 = n!$ possibilities.

2. How many ways can one choose k out of n items (order does matter)?

   $$\boxed{\frac{n!}{(n-k)!}}$$

   Use the same argument as before, except stop just before the $k+1$st element. Then you get $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!}$ possibilities.

3. How many ways can one choose k out of n items (order doesn't matter)?

   $$\boxed{\frac{n!}{k!(n-k)!}}$$

   This is the definition of combination, so it is clearly $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. This can be easily derived; there are n choices for the first item, $(n-1)$ for the second, until eventually there are $(n-(k-1))$ for the kth. However, there are $k!$ orderings of these k items, there must be $\frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$ total combinations.

4. How many sequences $a_1, a_2, \ldots, a_k$ of positive integers exists such that $a_i < a_{i+1} \leq n$?

   $$\boxed{\binom{n}{k}}$$

   Clearly, each of the a_i is distinct. Also, given any k distinct numbers, there will be exactly 1 sequence of a_i with those k numbers in it that satisfies the requirement. This creates a bijection[1] between sequences $a_1, \ldots, a_k$ and sets of k numbers chosen from the numbers 1,2,...,n. The number of sets for the latter is clearly $\binom{n}{k}$, so there are $\binom{n}{k}$ such sequences.

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[1] An over-simplified explanation of a bijection is that it is a one-to-one relationship between two sets created by a function. Their main use in combinatorics is that the sizes of the two sets must be the same, so if one set's size is known, both sizes are known.

5. How many ways can n distinct items be partitioned into k groups?

   $\boxed{n^k}$

   Consider each item individually. For each item, there are k options of what group to put it in. There are n such choices, each of which is independent, so there are n^k ways to perform the partition.

6. How many ways can one travel from $(0,0)$ to (x,y) traveling only to the right and up and only between adjacent lattice points.

   $\boxed{\binom{x+y}{x}}$

   Consider representing each path with a list of what direction was travelled, either right (R) or up (U). Clearly, any path can be represented by a distinct arrangement of the sequence $RR\cdots RUU\cdots U$, where there are x Rs and y Us. This creates a bijection, so the number of paths is the same as the number of arrangements, which is $\binom{x+y}{x}$.

7. How many ways can n distinct items be partitioned into k groups of sizes $s_1, s_2, \ldots, s_k$, where $s_1 + \ldots + s_k = n$?

   $\boxed{\frac{n!}{s_1!s_2!\cdots s_k!}}$

   Consider selecting each group seperately. For the first group, there are $\binom{n}{s_1}$ options, for the second, $\binom{n-s_1}{s_2}$, etc. Multiplying these out, there is a total of $\binom{n}{s_1}\binom{n-s_1}{s_2}\binom{n-s_1-s_2}{s_3}\cdots = \frac{n!}{s_1!s_2!\cdots s_k!}$. This value is often expressed as $\binom{n}{s_1, s_2, \ldots, s_k} = \frac{n!}{s_1!s_2!\cdots s_k!}$, and is refered to as *multinomial* coefficients. Note that these have a relationship with the term $a_1^{s_1}\cdots a_k^{s_k}$ in the expansion of $(a_1 + \ldots + a_k)^n$ similar to that of binomial coefficients to the expansion of $(x+y)^n$. Also note that if $s_1 + \ldots + s_k < n$, one can add a pseudo-set s_{k+1} which contains all of the items not in any of the first k sets (so its size will be $n - (s_1 + \ldots + s_k)$), and calculate the coefficient from there. Note that this value is also the number of ways to arrange n objects in k indistinct groups of sizes $s_1, s_2, \ldots, s_k$, where the sizes again sum to n.

8. Simplfify $\binom{n}{k} + \binom{n}{k+1}$.

   $\boxed{\binom{n+1}{k+1}}$

   Method one. Writing out the factorial notation and simplifying quickly yields the desired reuslt. $\frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} = \frac{n!(k+1)}{(k+1)!(n-k)!} + \frac{n!(n-k)}{(k+1)!(n-k)!} = \frac{n!(n+1)}{(k+1)!(n-k)!} = \binom{n+1}{k+1}$. This is the basis of Pascal's Triangle.

   Method two. Consider selecting $k+1$ items from $n+1$. Either the first item is chosen, in which case there are $\binom{n}{k}$ ways to select the remaining k, or the first item is not chosen, in which case there are $\binom{n}{k+1}$ ways to choose the remaining $k+1$. Thus, $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$.

9. Simplfify $\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n-1} + \binom{n}{n}$.

   $\boxed{2^n}$

   This can be derived easily by counting in two ways. This is the total number of subsets of a set of size n; for each element of the original set, there are two options: in the set, or not in the set. Thus, there are 2^n subsets, and the expression above simplifies to 2^n.

10. How many positive integer solutions exist to the system $a_1 + a_2 + \ldots + a_k = n$?

    $\boxed{\binom{n-1}{k-1}}$

Draw n circles in a row, and consider drawing $k-1$ lines in the $n-1$ gaps between the circles. Let a_1 be the number of circles before the first line, a_2 be the number of circles between the first and second lines, etc. Clearly, there will be k total a_i (if there are $k-1$ lines), all a_i will be at least 1 (since there is at least 1 circle between each gap) and the sum of the a_i will be n (since each circle is counted exactly once). Thus, there is a bijection between the sequence $a_1, \ldots, a_k$ and the drawing; as the drawing was generated by choosing $k-1$ lines from $n-1$ gaps, there are $\binom{n-1}{k-1}$ drawings, and thus $\binom{n-1}{k-1}$ sequences. This algorithm is sometimes refered to as "Stars and Bars."

11. How many non-negative integer solutions exist to the system $a_1 + a_2 + \ldots + a_k = n$?

$$\boxed{\binom{n+k-1}{k-1}}$$

Draw $n+k-1$ circles in a row, and cross out $k-1$ of them. Let a_1 be the number of circles before the first crossed-out one, a_2 be the number of circles between the first crossed-out one and the second crossed-out one, etc. Clearly, there will be k total a_i (if there are $k-1$ crossed out circles), all a_i will be at least 0 (since there may be two consecutive crossed-out circles) and the sum of the a_i will be n (since each remaining circle is counted exactly once, and there are $(n+k-1)-(k-1) = n$ remaining circles). Thus, there is a bijection between the sequence $a_1, \ldots, a_k$ and the drawing; as the drawing was generated by choosing $k-1$ circles from $n+k-1$ choices, there are $\binom{n+k-1}{k-1}$ drawings, and thus $\binom{n+k-1}{k-1}$ sequences. Note that this algorithm can be used to determine the number of ways to partition n indistinct objects into k sets.

12. How many integer solutions exist to the system $a_1 + a_2 + \ldots + a_k = n$ if $a_i > m$?

$$\boxed{\binom{n-km-1}{k-1}}$$

Method one. Use algebraic manipulation. Subtract km from both sides, so that now we have $b_1 + b_2 + \ldots + b_k = n - km$ with $b_i > 0$ (we subtracted m from each of the a_i's). Now apply the previous argument and arrive at $\binom{n-km-1}{k-1}$.

Method two. Use a bijection. This is the same as dividing up a set of size n into k groups, all of size greater than m. But this means we can remove m items from each group and still have a positive number of items in each group, yielding a scenario in which we divide up the set of size $n-km$ into k non-empy groups. Also, for every scenario with this smaller set we can add m items to each group to get back to what we had in the larger set. This means there is a one-to-one correspondence between the number of possibilities in both cases. We count the easier quantity (the set of size $n-km$). To ensure that all sets are non-empty, we simply place $k-1$ dividing lines between elements of the set to divide the set into k subsets. There are $\binom{n-km-1}{k-1}$ ways to do this, as there are $n-km-1$ spaces between successive elements of the set.

13. How many ways can one arrange n of one object and k of another if none of the k objects may be next to each other?

$$\boxed{\binom{n+1}{k}}$$

First, lay out the first objects in a row of length n. There are now $n+1$ gaps to place the k other objects in (there are $n-1$ gaps between the layed-out objects and 1 on both ends of the row). Thus, there are $\binom{n+1}{k}$ arrangements of these objects to avoid having any of the k objects touch each other.

14. How many ways can one place k indistinct items between n other indistinct items? (Any number of the first type can go between the sucessive items of the second type).

$$\boxed{\binom{n+k-2}{k}}$$

Draw $n+k$ circles in a row, and mark k of them as objects of the second type. Clearly, one cannot mark either the first or the last, as the problem specifies that the second objects go between objects of the first type. Thus, there are $n+k-2$ markable objects, and k marks, so there are $\binom{n+k-2}{k}$ arrangements.

15. Simplify $\binom{n}{0}+\binom{n+1}{1}+\ldots+\binom{n+k}{k}$.

$$\boxed{\binom{n+k+1}{k}}$$

The proof of this is trivial by induction. The base case is $k=1$, and it is clear that $\binom{n}{0}+\binom{n+1}{1}=1+n+1=\binom{n+2}{1}$. For the induction, $\binom{n}{0}+\binom{n+1}{1}+\ldots+\binom{n+k-1}{k-1}+\binom{n+k}{k}=\binom{n+k}{k-1}+\binom{n+k}{k}=\binom{n+k+1}{k}$. This is known as the *Hockey-stick Identity*, due to its appearance when drawn on Pascal's Triangle. Note that replacing $\binom{n}{i}$ with $\binom{n}{n-i}$ throughout the problem yields an alternative form of the identity.

16. How many paths of length n can be made using only left or right moves of length 1, starting on the left side of a line segment of length n?

$$\boxed{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$$

Consider the number of ways one can return to the origin with a path of length $2n$ (note that there is clearly no way to return to the origin with a path of length $2n+1$); let this value be $g(n)$. Clearly, the first move and the last move are preordained; the first move is invariably a move to the right and the last move a move to the left. Thus, if the path never returns to the origin until the last move, there are $g(n-1)$ options. If the path returns on the second move, then there are $g(n-2)$ options; on the fourth move, $g(1)g(n-3)$, etc. Thus, there are $g(n)=g(n-1)+g(n-2)+g(1)g(n-3)+g(2)g(n-4)+\ldots+g(n-4)g(2)+g(n-2)g(1)+g(n-2)+g(n-1)$ options. This is a well-known recurrance relation for the Catalan numbers, so $g(n)$ is the nth Catalan number, which is $\frac{(2n)!}{(n)!(n+1)!}$.

Now, consider $f(n)$ to be the number of paths possible after n moves (without the return to the origin restriction placed on g.) A recurrance relation can be easily found; any path of length $x-1$ that does not end at the origin has two child paths of length x, and any path of length $x-1$ that does end at the origin has one child path of length x. Thus, for even numbers $(2n+2)$, the recurrance is $f(2n+2)=2f(2n+1)$, as no odd path can end at the origin. For odd numbers $(2n+1)$, the recurrance is $f(2n+1)=2f(2n)-g(n)$.

The proof will be completed through induction. Assume the solution is $f(n)=\binom{n}{\lfloor \frac{n}{2} \rfloor}$. The base cases are $n=1 \rightarrow f(n)=1$ and $n=2 \rightarrow f(n)=2$, so the base case is complete. For any odd number $2n+1$, $f(2n+1)=2f(2n)-g(n)$. Testing this, $2f(2n)-g(n)=2\binom{2n}{\lfloor \frac{2n}{2} \rfloor}-\frac{1}{n+1}\binom{2n}{n}=2\binom{2n}{n}-\frac{1}{n+1}\binom{2n}{n}=\binom{2n}{n}(2-\frac{1}{n+1})=\binom{2n}{n}(\frac{2n+1}{n+1})=\binom{2n+1}{n+1}=\binom{2n+1}{n}=f(2n+1)$, so the induction works for all $2n+1$. For any even number $2n+2$, $f(2n+2)=2f(2n+1)$. Testing this in the induction, $2f(2n+1)=2\binom{2n+1}{\lfloor \frac{2n+1}{2} \rfloor}=2\binom{2n+1}{n}=\binom{2n+1}{n}+\binom{2n+1}{n+1}=\binom{2n+2}{n+1}=f(2n+2)$, so the induction works for all $2n+2$. Thus, the induction is complete, and $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ is the number of paths possible.

17. How many ways can one place in order a total of n elements of two different types if one can at no point have placed down more elements of the first type than of the second type?

$$\boxed{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$$

If one considers placing items of the second type moves to the right and items of the first type moves to the left, there is a clear bijection to the scenario of the previous problem.