Elementary Combinatorics

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1 Basics

Combinatorics, or "the study of finite collections of objects,"[1] is a often-covered topic on math contests. To be fully prepared for any major contest, a basic grasp of combinatorial techniques is needed. However, many such techniques are difficult to describe, and can only be discovered through solving many different types of problems. The goal of this lecture is to introduce the basics of combintorics, so that the advanced concepts needed can be learned through practice.

1.1 Factorials

A basic concept used throughout combinatorics is the factorial. This function, denoted with an exclamation point, is recursively defined for whole numbers as:

$$n! = n * (n-1)!$$

$$0! = 1$$

or, non-recursively

$$n! = \prod_{i=1}^{n} i$$

The factorials up to 7! occur often and are worth memorizing.

$$\begin{aligned} 0! &= 1 \\ 1! &= 1 \\ 2! &= 2 \\ 3! &= 6 \\ 4! &= 24 \end{aligned}$$

[1] Wikipedia

$$\begin{aligned}
5! &= 120\\
6! &= 720\\
7! &= 5040\\
8! &= 40320\\
9! &= 362880\\
10! &= 3628800
\end{aligned}$$

Factorials are used to find the number of ways to order any given number of objects. For example, given n books with which to place on a shelf, there are $n!$ ways to do so, as there will be n books that could go in the first spot, $(n-1)$ that could go in the second spot, etc., until there are 2 options for the $(n-1)$ spot and 1 option for the n spot.

1.2 Independent Events

To expand combinatorics into multiple events, a fundamental part of probability and combinatorics is needed: the idea that two events are *independent*, that is, the outcome of one does not effect the outcome of the other. Where a and b are independent events, and N is the number of outcomes from a certain event, then $N(a\&b) = N(a)N(b)$. This is a bit easier to understand with a demonstration.

Consider a standard license plate problem: Given a license plate which has 3 letters followed by 4 numbers, how many potential license plates are there? Here, there are 7 events; call them l_1, l_2, l_3, n_1, n_2, n_3, and n_4. These events are independent, and $N(l_i) = 26$, $N(n_i) = 10$. thus, $N(l_1\& \ldots \& n_4) = N(l_1) * \ldots * N(n_4) = 26 * 26 * 26 * 10 * 10 * 10 * 10 = 26^3 10^4 = 175760000$.

1.3 Combinations and Permutations

When selecting a group of objects from a larger group, there are two ways to do so. In one, the order of the objects chosen does not matter; this is a combination. In the other, order is important; this is a permutation. The formulas for combinations and permutations reflect this.

${}_nP_r$ (read n permute r) is a permutation of r objects from a set of n objects. To find the equation for ${}_nP_r$, one realizes that for the first object, there will be n choices, for the second, $(n-1)$, until for the rth object, there are $(n-r+1)$ choices. Thus:

$${}_nP_r = (n)(n-1)\ldots(n-r+1) = \frac{(n)(n-1)\ldots(2)(1)}{(n-r)(n-r-1)\ldots(2)(1)} = \frac{n!}{(n-r)!}$$

$\binom{n}{r}$ (read n choose r), is a combination of r objects from a set of n. The objects can be selected the same was as they are for ${}_nP_r$, but once the set of r objects has been chosen, there are $r!$ arrangements of this set. With combinations, these arrangements are all considered to be the same, so it is neccessary to divide the ${}_nP_r$ formula by $r!$ to find the formula for $\binom{n}{r}$.

$$\binom{n}{r} = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!}$$

1.4 Pascal's Triangle

Pascal's Triangle is a famous mathematical shape, which is formed by letting any given number in it be equal to the sum of the 2 numbers above it.

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

Its application to combinatorics is due to the patterns found in Pascal's Triangle. Note that row n will have $n+1$ columns in it (assuming the first row is row 0). If each number is assigned a column, with the first number in each row being given column number 0 (and thus the last being given column number n), then the number in row r and column c is $\binom{r}{c}$.

1.5 Binomial Theorem

The Binomial Theorem allows large powers of binomials to be written out in a relatively clean fashion. For positive n,

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

For example, $(x+y)^4 = \sum_{i=0}^{4} \binom{4}{i} x^i y^{4-i} = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$. Note that the coefficients of the polynomial are the same as those across Pascal's Triangle.

Defining $0! = 1$ makes much more sense now, as this allows the binomial theorem to be written out without ugly cases.

2 Advanced Concepts

2.1 Inclusion/Exclusion

To understand Inclusion/Exclusion, a brief introduction to set notation is needed. $|A|$ indicates the size of A, or the number of elements that A contains. $A \bigcup B$ (read A union B or

A cup B) is the set of elements contained in A or B or both. $A \bigcap B$ (read A intersection B or A cap B) is the set of elements contained in both A and B.

Inclusion/Exclusion is a formal method for solving Venn diagram problems. For the case with only two sets, this concept states that:

$$|A \bigcup B| = |A| + |B| - |A \bigcap B|$$

For the three set case, Inclusion exclusion claims:

$$|A \bigcup B \bigcup C| = |A| + |B| + |C| - |A \bigcap B| - |B \bigcap C| - |A \bigcap C| + |A \bigcap B \bigcap C|$$

This can be generalized[2] as follows:

$$|A_1 \bigcup A_2 \bigcup \ldots \bigcup A_p| = \sum_{1 \le i \le p} |A_i| - \sum_{1 \le i_1, i_2 \le p} |A_{i_1} \bigcap A_{i_2}| + \ldots + (-1)^{p-1} |A_1 \bigcap A_2 \bigcap \cdots \bigcap A_p|$$

Note that sometimes, you will be given A, $A \bigcup B$, $A \bigcap B$ and asked to find B. Here, you can solve for B using Inclusion/Exclusion, which will probably be faster than solving for B directly.

2.2 Combinatorial Identities

Problems that involve large amounts of counting can often be simplified by judicious application of a few simple combinatorial identities.

Note that (2) provides for the formal definition of Pascal's Triangle, and (3) can be proven by expanding $(1-1)^n$ using the Binomial Theorem.

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n \quad (1)$$

$$\binom{n}{i} + \binom{n}{i+1} = \binom{n+1}{i+1} \quad (2)$$

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i = 0 \quad (3)$$

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n} \quad (4)$$

[2]Mathworld

2.3 Diophantine Equations

Often, one will encounter problems of the following form:

$$x_1 + x_2 + \ldots + x_n = y \quad x_i \in \mathbb{Z}^+$$

or

$$x_1 + x_2 + \ldots + x_n = y \quad x_i \in \mathbb{Z}^+ \cup 0$$

where the problem asks how many n-tuples $(x_1, x_2, \ldots, x_n)$ exist. These can be solved using combinatorics!

For the first problem, consider $(y + n - 1)$ objects in a row. $(n - 1)$ of these objects will then be crossed out. Then, x_1 will be assigned the number of objects between the start and the first crossed out item, x_2 will be assigned the numebr of objects between the 1st and 2nd crossed out objects, etc. This creates a *bijection* between the ordered pairs and these object diagrams. Thus, the numebr of ordered pairs is the same as the number of diagrams, which is the number of ways to choose $(n - 1)$ objects to cross out from $(y + n - 1)$ objects, which is $\binom{y+n-1}{n-1}$.

The second problem is exactly the same, but this time, the restriction requires positive integers rather than non-negative integers. Here, the diagram is y objects in a row. $(n - 1)$ lines will then be drawn in the $(y - 1)$ gaps between the objects (with at most 1 line in each gap). This again creates the bijection between the diagrams and the ordered pairs, indicating that the number of ordered pairs is $\binom{y-1}{n-1}$.

3 Problems

1. (AHSIMC 2005) What is the coefficient of the fourth term of $(x+y)^{15}$?

2. (DSchafer05) Assume that of 1600 students at Thomas Jefferson, 100 are on math team, 50 are on computer team, and 20 are on both. How many students at TJ do not do math team or computer team?

3. (DSchafer05) At least 2 members of a group of 12 people will go out for pizza. How many different pizza groups are possible?

4. (DSchafer06) How many solutions are there to the equation $w+x+y+z=13$, where w, x, y, z are positive integers?

5. (HMMT Combinatorics 2005, #3) The Red Sox play the Yankees in a best-of-seven series that ends as soon as one team wins four games. Suppose that the probability that the Red Sox win Game n is $\frac{n-1}{6}$. What is the probability that the Red Sox will win the series?

6. (HMMT Combinatorics 2004, #3) A class of 10 students took a math test. Each problem was solved by exactly 7 of the students. If the first nine students each solved 4 problems, how many problems did the tenth student solve?

7. (HMMT Combinatorics 2003, #3) Daniel and Scott are playing a game where a player wins as soon as he has two points more than his opponent. Both players start at par, and points are earned one at a time. If Daniel has a 60% chance of winning each point, what is the probability that he will win the game?

8. (AHSME 1989 #29) Determine the value of $\displaystyle\sum_{k=0}^{49}(-1)^k\binom{99}{2k}$.

4 Hints

All of the these hints are written in order to avoid giving away the solution, but it is still better to try each problem for a reasonable amount of time before using these hints.

1. Try rereading the section on the Binomial Theorem.
2. Try rereading the section on Inclusion/Exclusion.
3. Although this problem is solvable by brute force, using a combinatorial identity makes it a lot easier. Consider some of the concepts of Inclusion/Exclusion.
4. Try rereading the section on Diophantine Equations.
5. Consider what would happen if the series did not stop when a team reached 4 wins. Now what is the probability that the Red Sox win?
6. Consider how many problems were solved in total by the class.
7. Assuming that Daniel and Scott both win 1 point, now what is the probability that Daniel will win?
8. Consider the binomial expansions of $(1+i)^x$ and $(1-i)^x$.

5 Solutions

It is highly recommended that you not look at the solutions until you have put forth a strong effort to solve the problems yourself.

1. $\boxed{\mathbf{455}}$ The 4th coefficient will be $\binom{15}{3} = \frac{15!}{12!3!} = \frac{15*14*13}{3*2*1} = 455$.

2. $\boxed{\mathbf{1470}}$ $1600 - 100 - 50 + 20 = 1470$.

3. $\boxed{\mathbf{2035}}$ The number of groups is equal to $\displaystyle\sum_{i=2}^{12}\binom{12}{i}$. The problem can be solved by brute force, but remembering that $\displaystyle\sum_{i=0}^{n}\binom{n}{i} = 2^n$ with $n = 12$ allows the more elegant solution $\displaystyle\sum_{i=2}^{12}\binom{12}{i} = \sum_{i=0}^{12}\binom{12}{i} - \binom{12}{0} - \binom{12}{1} = 2^{12} - 1 - 12 = 2035$.

4. $\boxed{\mathbf{110}}$ Draw the diagram, with 13 objects. There are 12 gaps, into which one must draw three lines to create the bijection between the diagrams and the 4-tuples, so there are $\frac{12*11*10}{3*2*1} = 110$

5. $\boxed{\mathbf{\frac{1}{2}}}$ Assuming all of the the series are played out to their conclusions, the Red Sox will win exacly $\frac{1}{2}$ of the series, as the Red Sox are just as likely to win game k as the Yankees are to win game $8 - k$. As playing out a series to its conclusion will never change the result of the series (once a team has 4 wins, it cannot possibly lose), this approach is valid and yields the correct answer.

6. $\boxed{\mathbf{6}}$ Suppose the last student solved n problems, and the total number of problems on the test was p. Then the toal number of correct solutions written was $7p$ (seven per problem), ans also equal to $36 + n$ (the sum of the students' scores), so $p = \frac{36+n}{7}$. The smallest $n \geq 0$ for which this is an integer is $n = 6$. But we must also have $n \leq p$, so $7n \leq 36 + n$. and solving gives $n \leq 6$. Thus $n = 6$ is the answer.

7. $\boxed{\mathbf{\frac{9}{13}}}$ Consider the situation after two points. Daniel has a $\frac{9}{25}$ chance of winning, Scott, $\frac{4}{25}$ and there is a $\frac{12}{25}$ chance that the players will be tied. If we define the proability that Daniel wins to be p, then $p = \frac{9}{25} + \frac{12}{25}p$, and $p = \frac{9}{13}$.

8. $\boxed{\mathbf{-2^{49}}}$ The given expression is equal to $\frac{(1+1)^{99}+(1-i)^{99}}{2}$. Using DeMoivre's Theorem easily simplifies this expression to -2^{49}.