Root Equations and Recursion

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October 7, 2005

1 Root Equations

1.1 Basic Concept

The basic concept of root equations is that an infinite function can usually be expressed as an equation $x = f(x)$. By substituting, it is clear that this equation is equivelent to $x = f(f(f(\ldots f(x)\ldots)))$. This fact can be used to solve different generated equations.

1.2 Simple Root Equations

Consider the following equation:

$$x = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots}}}}$$

How can this be solved? Cosider squaring the left side, then suptracting 2 from both sides.

$$x^2 - 2 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots}}}}$$

The right side is the same as it was before! Thus,

$$x^2 - 2 = x$$

$$x = 2, -1$$

-1 is quickly eliminated as an extraneous solution, as it results in $-1 = \sqrt{1}$. Thus, the only solution is $x = 2$.

*With much material stolen from Mildorf04

Consider this example:

$$x = \frac{2}{\frac{2}{x} + 2 + \frac{x}{2}} + 2 + \frac{\frac{2}{x} + 2 + \frac{x}{2}}{2}$$

The problem can be started by turning the equation into a quartic:

$$x = \frac{2}{\frac{2}{x} + 2 + \frac{x}{2}} + 2 + \frac{\frac{2}{x} + 2 + \frac{x}{2}}{2}$$

$$x = \frac{4x}{x^2 + 4x + 4} = 2 + \frac{x^2 + 4x + 4}{4x}$$

$$3x^4 - 56x^2 - 64x - 16 = 0$$

Here, though, the only obvious course to take is guess-and-check, possibly by using the Rational Root Theorem. Surely there is a better way? Examining the original problem, it is clear that this was generated using a root equation, and that the root equation used can be solved to find some solutions.

$$x = \frac{2}{x} + 2 + \frac{x}{2}$$

$$x^2 - 4x - 4 = 0$$

$$x = \frac{4 \pm \sqrt{32}}{2} = 2 \pm 2\sqrt{2}$$

Now, $x^2 - 4x - 4$ can be factored out of the quartic.

$$(x^2 - 4x - 4)(3x^2 + 12x + 4) = 0$$

$$3x^2 + 12x + 4 = 0$$

$$x = \frac{-12 \pm \sqrt{96}}{6} = -2 \pm \frac{2\sqrt{6}}{3}$$

Thus, the 4 solutions are $2 \pm 2\sqrt{2}$ and $-2 \pm \frac{2\sqrt{6}}{3}$.

1.3 Composition of Functions

Consider the equation $f(g(h(x))) = 0$. TO solve this, we can find all values $f_1, f_2, \ldots, f_n$ such that $f(f_i) = 0$, then solve $g(x) = f_i$, yielding solutions g_1, g_2 through g_n, and finally solve $h(x) = g_i$ to find all solutions for x.

2 Problems

3 Hints

All of the these hints are written in order to avoid giving away the solution, but it is still better to try each problem for a reasonable amount of time before using these hints.

4 Solutions

It is highly recommended that you not look at the solutions until you have put forth a strong effort to solve the problems yourself.