Basic Sequences and Series

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1 The Basics

Before we begin discussing sequences and series, we must first define them. A sequence is, simply put, a list of numbers. Sequences are most often symbolized as $a_1, a_2, a_3, \ldots$, where a_1 is the first entry in sequence a. A series is simply the sum of the sequence.

1.1 Arithmetic

This is a sequence such that $a_i - a_{i-1} = d$, where d is a constant difference. Arithmetic sequences are nice, because one can easily find pretty much any term. Given a term a_i and the common difference d, it should be reasonably clear that:

$$a_j = a_i + d(j - i)$$

If two terms (and not the difference) are given, this formula can also be used. We also like arithmetic sequences because they are easy to sum. Using the same notation as before,

$$a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n = \frac{n}{2}(a_1 + a_n)$$

If this doesn't make sense, consider that one can pair the first and last terms, the second and pentultimate terms, continuing on, and that each of these pairs will have the same sum.

Example 1 (FCML 2006 #7-3)

Given that $a, b, c, 2b$ is an arithmetic sequence, find $\frac{c}{a}$.

Solution

Rewrite the sequence $a, a + r, a + 2r, a + 3r$. Clearly, $\frac{2b}{b} = \frac{a+3r}{a+r} = 2$, and $a = r$. Thus, $\frac{c}{a} = \frac{a+2r}{a} = 3$.

1.2 Geometric

If arithmetic sequences are the easiest type to work with, then geometric is surely the second easiest. A geometric sequence is defines such that $\frac{a_i}{a_{i-1}} = r$, where r is a constant ratio. Again, finding any term in a geometric sequence is pretty easy:

$$a_j = r^{j-i} a_i$$

This can once again be used to find the common ratio if the terms are given without the ratio. Summing geometric sequences is also reasonably nice[1]:

$$a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n = a_1 + a_1 r + a_1 r^2 + \ldots + a_1 r^{n-1} = a_1 \frac{1 - r^n}{1 - r}$$

[1] This is a factorization you should be familiar with: $\frac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + \ldots + x + 1$.

If $|r| < 1$, then the infinite sum of the geometric series can be found:

$$a_1 + a_2 + a_3 + \ldots = \frac{a_1}{1-r}$$

This is clearly derived from the previous formula, as the r^n term approaches 0 as n approaches ∞ (**only** if $|r| < 1$: otherwise, the sum is infinite)

1.3 Recursive

A recursive sequence is one defined as such: $a_n = f(a_{n-1}, a_{n-2}...)$. The best example of a recursive sequence is the Fibonacci sequence, defined as $F_n = F_{n-1} + F_{n-2}$, where $F_1 = 1$ and $F_2 = 1$. Most of the recursive functions one encounters on math competitions are *linear* recursions, where the function f is a linear equation of a_{n-1}, a_{n-2}, and so on. For most of these functions, one can therefore solve for an explicit formula; normally, one will not have to do so, however, and the detailed explaination of this process is beyond the scope of this lecture. However, the explicit formula for the Fibonacci sequence is well worth memorizing:

$$F_n = \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

1.4 Misc.

There are many other facts about sequences one should know; a few are listed here:

$$\begin{aligned}
1 + 2 + \ldots + (n-1) + n &= \frac{n(n+1)}{2} \\
1^2 + 2^2 + 3^2 + \ldots + n^2 &= \frac{n(n+1)(2n+1)}{6} \\
1^3 + 2^3 + 3^3 + \ldots + n^3 &= \frac{n^2(n+1)^2}{4} \\
\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots &= \frac{\pi^2}{6}
\end{aligned}$$

2 What Now?

Although the techniques described in the above sections are useful, most of the problems one encounters on competitions require a bit more than simple formula memorization. There are a few techniques that allow more complex sequence problems to be solved.

2.1 Algebra

Try setting the sum to S: this very often simplifies the problem. For example, consider the following:

Example 2 (Traditional)

Find $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \ldots$

Solution

When one first looks at this problem, it is clear that the above formulae will not directly solve it: the sequence is not arithmetic or geometric. However, the numerators form an arithmetic sequence, and the denominators a geometric sequence, so if one could subtract adjacent terms, a geometric sequence would result! So, set the desired sum equal to S, and consider the following:

$$\begin{aligned}
S &= 2S - S \\
&= (\frac{1}{1} + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \ldots) - (\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \ldots) \\
&= \frac{1}{1} + (\frac{2}{2} - \frac{1}{2}) + (\frac{3}{4} - \frac{2}{4}) + (\frac{4}{8} - \frac{3}{8}) + \ldots \\
&= \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \\
&= \frac{1}{1 - \frac{1}{2}} \\
&= 2
\end{aligned}$$

The technique described above works in many situations like the above, where the sequence is not pretty, but the numerators form an arithmetic series. It also works very nicely with the Fibonacci sequence in the numerators, a fact which was the key to an HMMT 2006 Algebra problem.

Example 3 (HMMT 2006 - Algebra #4)

Find $\sum_{n=1}^{\infty} \frac{F_n}{4^{n+1}}$, where F_n is the nth Fibonacci number.

Solution

The first step when looking at virtually any sequence problem should be to write out the first few terms.

$$S = \frac{1}{4^2} + \frac{1}{4^3} + \frac{2}{4^4} + \frac{3}{4^5} + \frac{5}{4^6} + \frac{8}{4^7} + \ldots$$

What happens when S is divided by 4?

$$\frac{S}{4} = \frac{1}{4^3} + \frac{1}{4^4} + \frac{2}{4^5} + \frac{3}{4^6} + \frac{5}{4^7} + \ldots$$

$$S - \frac{S}{4} = \frac{1}{4^2} + \frac{1}{4^4} + \frac{1}{4^5} + \frac{2}{4^6} + \frac{3}{4^7} + \ldots$$

$$S - \frac{S}{4} = \frac{1}{16} + \frac{S}{16}$$

$$\frac{3S}{4} = \frac{S+1}{16}$$

$$S = \frac{1}{11}$$

Again, setting the sum equal to S and manipulating that allowed the sequence to turn into a pretty algebraic equation.

2.2 Telescoping

Telescoping is where we take a sum with many terms and cancel a large number of them out. In general:

$$\sum_{i=1}^{n} f(i) - f(i-1) = (f(1) - f(0)) + (f(2) - (f(1)) + \ldots + (f(n) - f(n-1)) = f(n) - f(0)$$

This can be used in many situations to solve problems involving complicated sequences. The easiest way to demonstrate telescoping is through examples.

Example 4 (Traditional)

Find $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\ldots$

Solution

This is clearly not arithmetic or geometric, and there is no easy way to make it so. However, an obvious telescope is: $(1-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+\ldots=1$

Example 5 (Traditional)

$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\frac{1}{\sqrt{4}+\sqrt{5}}+\ldots+\frac{1}{\sqrt{99}+\sqrt{100}}$$

Solution

Rationalizing each fraction quickly yields an easy telescope.

$$(\sqrt{2}-\sqrt{1})+(\sqrt{3}-\sqrt{2})+(\sqrt{4}-\sqrt{3})+(\sqrt{5}-\sqrt{4})+\ldots+(\sqrt{100}-\sqrt{99})$$

$$\sqrt{100}-\sqrt{1}=9$$

Example 6 (Mildorf)

$$\sum_{i=1}^{99}(k^2+k+1)k!$$

Solution

The key to this one is to factor creatively.

$$\begin{aligned}\sum_{i=1}^{99}(k^2+k+1)k! &= \sum_{i=1}^{99}\big((k+1)^2*k!-k*k!\big)\\ &= \sum_{i=1}^{99}((k+1)(k+1)!-k*k!)\\ &= 100*100!-1\end{aligned}$$