AHSME 1989 - Problem 29

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$$\sum_{k=0}^{49} (-1)^k \binom{99}{2k}$$

We will generalize the problem into:

$$\begin{aligned} S &= \sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k} \\ &= \binom{2n+1}{0} - \binom{2n+1}{2} + \binom{2n+1}{4} + \ldots + (-1)^k \binom{2n+1}{2k} + \ldots \end{aligned}$$

Now, consider the following 2 bionomial expansions:

$$\begin{aligned} (1+i)^{2n+1} &= \binom{2n+1}{0} + i\binom{2n+1}{1} - \binom{2n+1}{2} - i\binom{2n+1}{3} + \ldots \\ (1-i)^{2n+1} &= \binom{2n+1}{0} - i\binom{2n+1}{1} - \binom{2n+1}{2} + i\binom{2n+1}{3} + \ldots \end{aligned}$$

These can be used to solve for S as follows:

$$\begin{aligned} (1+i)^{2n+1} + (1-i)^{2n+1} &= 2\binom{2n+1}{0} - 2\binom{2n+1}{2} + \ldots \\ &= 2S \end{aligned}$$

$$
\begin{aligned}
S &= \frac{(1+i)^{2n+1} + (1-i)^{2n+1}}{2} \\
(1+i)^{2n+1} &= \left(\sqrt{2}\right)^{2n+1} \left(\cos\left(\frac{\pi\,(2n+1)}{4}\right) + i\sin\left(\frac{\pi\,(2n+1)}{4}\right)\right) \\
(1-i)^{2n+1} &= \left(\sqrt{2}\right)^{2n+1} \left(\cos\left(\frac{\pi\,(2n+1)}{4}\right) - i\sin\left(\frac{\pi\,(2n+1)}{4}\right)\right) \\
S &= \left(\sqrt{2}\right)^{2n+1} \left(\cos\left(\frac{\pi\,(2n+1)}{4}\right)\right)
\end{aligned}
$$

The case asked for in the problem is $n = 49$.

$$
\begin{aligned}
S &= \left(\sqrt{2}\right)^{99} \left(\cos\left(\frac{99\pi}{4}\right)\right) \\
&= \left(\sqrt{2}\right)^{99} \left(-\frac{\sqrt{2}}{2}\right) \\
&= -2^{49}
\end{aligned}
$$