Chapter 9 Sorting Algorithms

- rearranging a list of numbers into increasing (strictly non-decreasing) order.

Potential Speedup

$O(n\log n)$ is optimal for any sequential sorting algorithm without using special properties of the numbers.

Best we can expect based upon a sequential sorting algorithm but using n processors is

$$\text{Optimal parallel time complexity} = \frac{O(n\log n)}{n} = O(\log n)$$

Has been obtained by Leighton (1984) based upon an algorithm by Ajtai, Komlós, and Szemerédi (1983), but the constant hidden in the order notation is extremely large. Also an algorithm for an n-processor hypercube using random operations.

But, in general, a realistic $O(\log n)$ algorithm with n processors is a goal that will not be easy to achieve. It may be that the number of processors will be greater than n.

Rank Sort

The number of numbers that are smaller than each selected number is counted.

This count provides the position of selected number in sorted list; that is, its “rank.”

First `a[0]` is read and compared with each of the other numbers, `a[1]` … `a[n-1]`, recording the number of numbers less than `a[0]`.Suppose this number is x. This is the index of the location in the final sorted list.

The number `a[0]` is copied into the final sorted list `b[0]` … `b[n-1]`, at location `b[x]`.

Actions repeated with the other numbers.

Overall sequential sorting time complexity of $O(n^2)$ (not exactly a good sequential sorting algorithm!).

Sequential Code

```
for (i = 0; i < n; i++) {   /* for each number */
    x = 0;
    for (j = 0; j < n; j++) /* count number of nos less than it */
      if (a[i] > a[j]) x++;
    b[x] = a[i];/* copy number into correct place */
}
```

This code will fail if duplicates exist in the sequence of numbers.

Parallel Code
Using *n* Processors

One processor allocated to one of the numbers Processor finds the final index of one numbers in O(n) steps. With all processors operating in parallel, the parallel time complexity O(n).

In `forall` notation, the code would look like

```
forall (i = 0; i < n; i++) {/* for each number in parallel*/
  x = 0;
  for (j = 0; j < n; j++) /* count number of nos less than it */
    if (a[i] > a[j]) x++;
  b[x] = a[i];            /* copy number into correct place */
}
```

Parallel time complexity, O(n), is better than any sequential sorting algorithm.

We can do even better if we have more processors.

Using n^2 Processors

Comparing one number the other numbers in list performed using multiple processors:

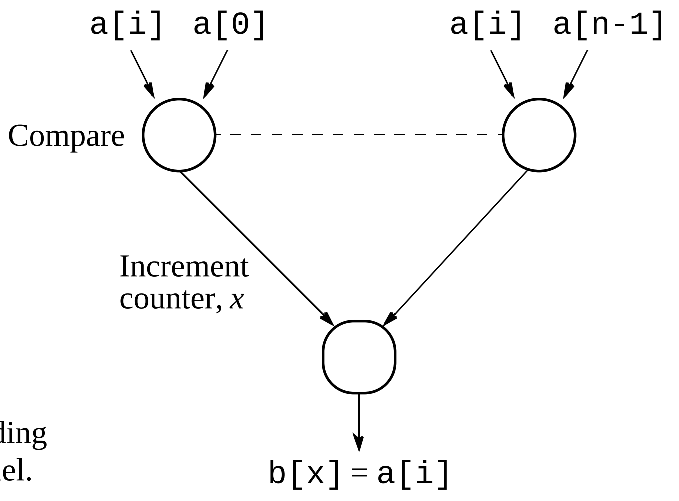


Figure 9.1 Finding the rank in parallel.

$n - 1$ processors are used to find the rank of one number. With n numbers, $(n - 1)n$ processors or (almost) n^2 processors needed. Incrementing the counter is done sequentially and requires a maximum of n steps. Total number of steps is by $1 + n$.

Reduction in Number of Steps

Tree structure to reduce the number of steps involved in incrementing the counter:

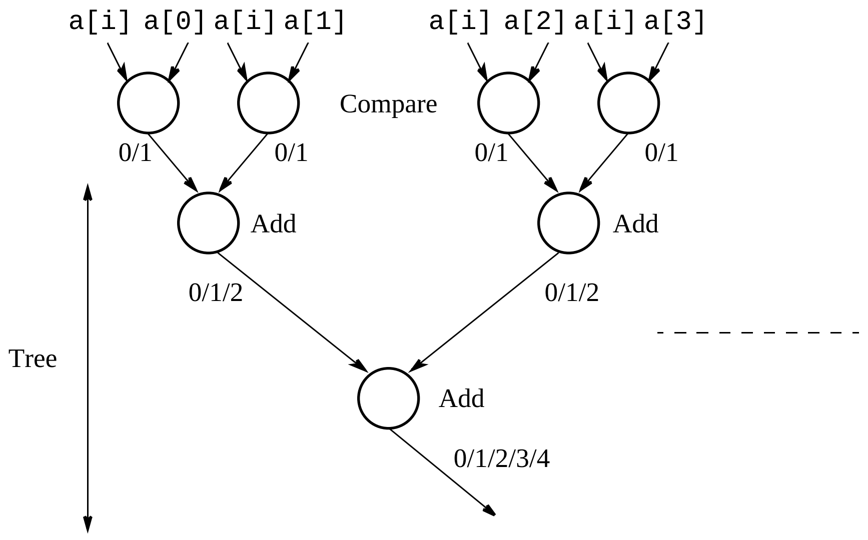


Figure 9.2 Parallelizing the rank computation.

O($\log n$) algorithm with n^2 processors. Processor efficiency relatively low.

Parallel Rank Sort Conclusions

Rank sort can sort in O(n) with n processors or in O($\log n$) using n^2 processors.

In practical applications, using n^2 processors will be prohibitive.

Theoretically possible to reduce time complexity to O(1) by considering all increment operations as happening in parallel since they are independent of each other. O(1) is, of course, the lower bound for any problem.

Message Passing Parallel Rank Sort

Master-Slave Approach

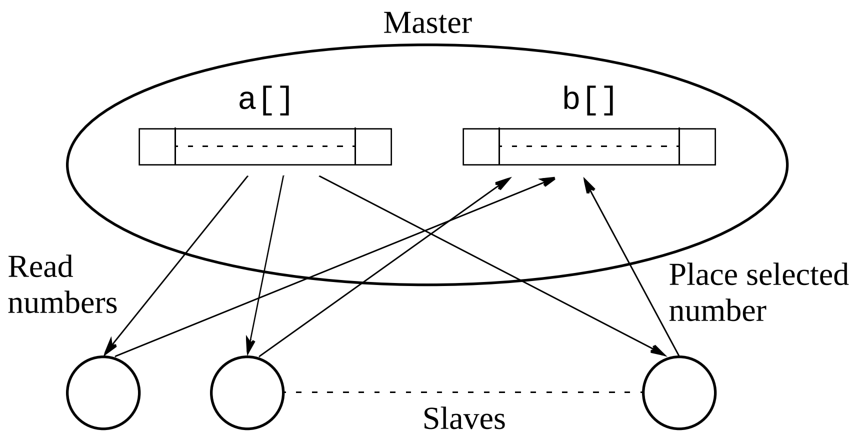


Figure 9.3 Rank sort using a master and slaves.

Requires shared access to the list of numbers. Master process responds to request for numbers from slaves. Algorithm better for shared memory

Compare-and-Exchange Sorting Algorithms

Compare and Exchange

Form the basis of several, if not most, classical sequential sorting algorithms.

Two numbers, say *A* and *B*, are compared. If *A* > *B*, *A* and *B* are exchanged, i.e.:

```
if (A > B) {
  temp = A;
  A = B;
  B = temp;
}
```

Message-Passing Compare and Exchange

To implement compare and exchange is for P_1 to send A to P_2, which compares A and B and sends back B to P_1 if A is larger than B (otherwise it sends back A to P_1):

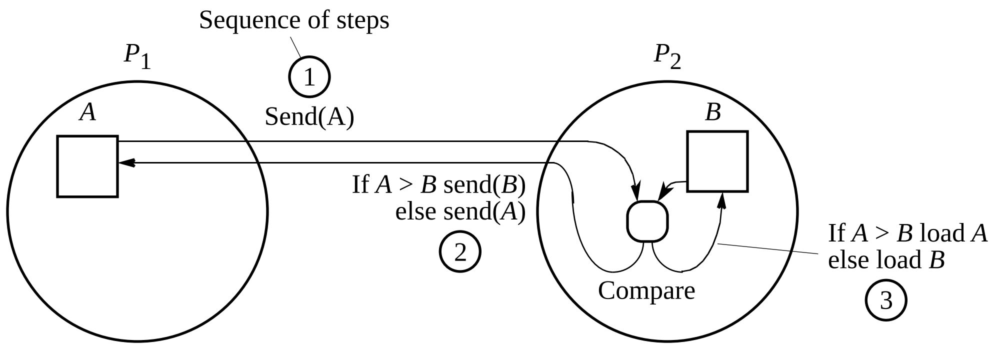


Figure 9.4 Compare and exchange on a message-passing system — Version 1.

Code:

Process P_1

```
send(&A, P2);
recv(&A, P2);

Process P2

recv(&A, P1);
if (A > B) {
  send(&B, P1);
  B = A;
} else
  send(&A, P1);
```

Alternative Message Passing Method

For P_1 to send A to P_2 and P_2 to send B to P_1. Then both processes perform compare operations. P_1 keeps the larger of A and B and P_2 keeps the smaller of A and B:

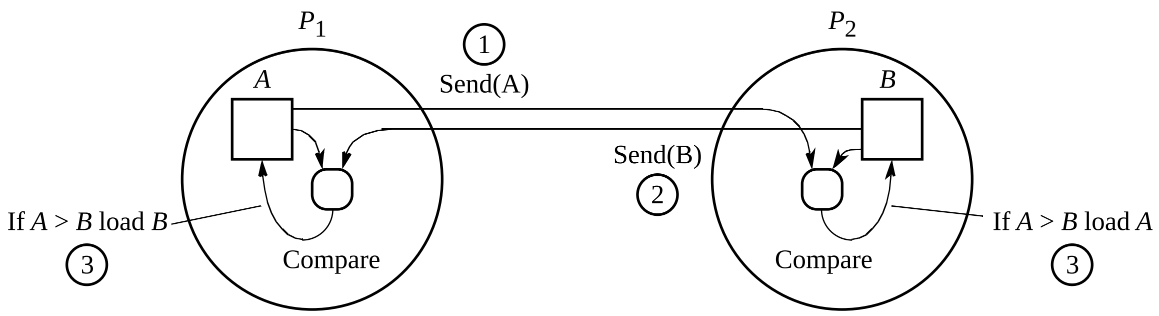


Figure 9.5 Compare and exchange on a message-passing system — Version 2.

Code:

Process P_1

```
    send(&A, P2);
    recv(&B, P2);
    if (A > B) A = B;
```

Process P_2

```
    recv(&A, P1);
    send(&B, P1);
    if (A > B) B = A;
```

Process P_1 performs the `send()` first and process P_2 performs the `recv()` first to avoid deadlock. Alternatively, both P_1 and P_2 could perform `send()` first if locally blocking (asynchronous) sends are used and sufficient buffering is guaranteed to exist - *not safe message passing.*

Note on Precision of Duplicated Computations

Previous code assumes that the `if` condition, `A > B`, will return the same Boolean answer in both processors.

Different processors operating at different precision could conceivably produce different answers if real numbers are being compared.

This situation applies to anywhere computations are duplicated in different processors to reduce message passing, or to make the code SPMD.

Data Partitioning

p processors and *n* numbers. A list of *n*/*p* numbers would be assigned to each processor:

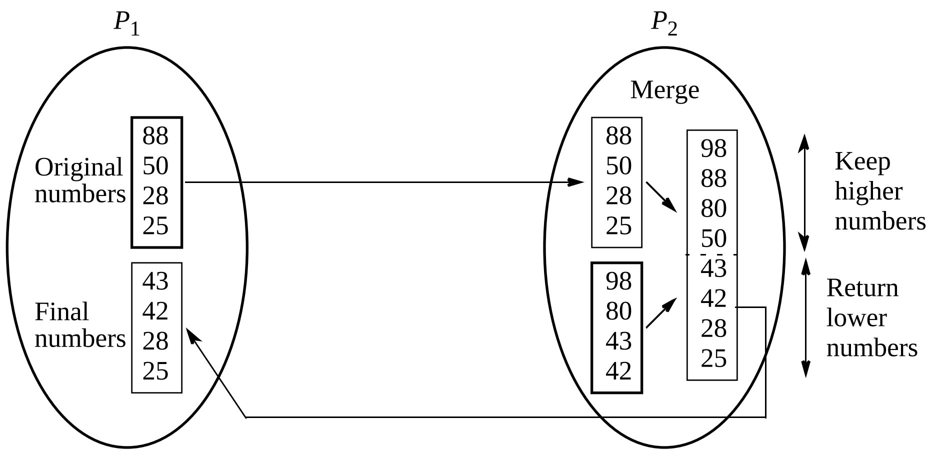


Figure 9.6 Merging two sublists — Version 1.

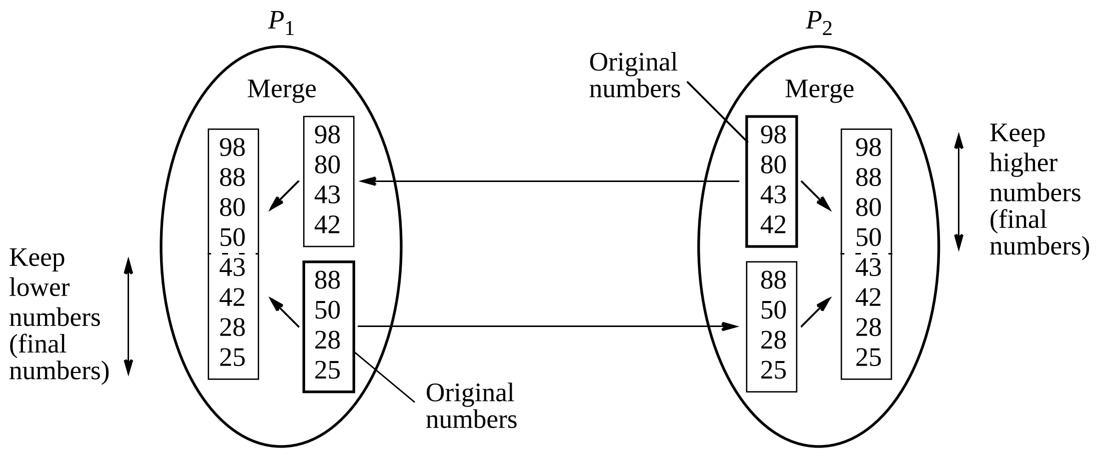


Figure 9.7 Merging two sublists — Version 2.

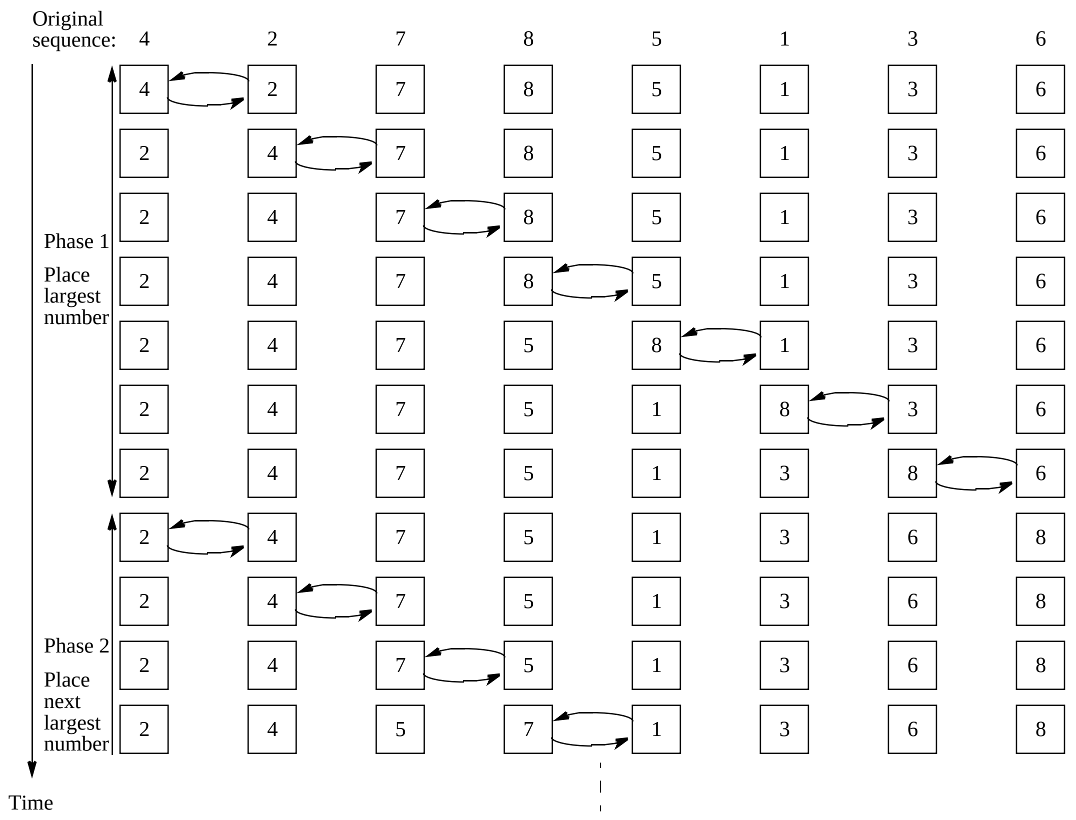


Figure 9.8 Steps in bubble sort.

Sequential Code

With numbers held in array `a[]`:

```
for (i = n - 1; i > 0; i--)
  for (j = 0; j < i; j++) {
    k = j + 1;
    if (a[j] > a[k]) {
      temp = a[j];
      a[j] = a[k];
      a[k] = temp;
    }
  }
```

Time Complexity

$$\text{Number of compare and exchange operations} = \sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$$

which indicates a time complexity of $O(n^2)$ given that a single compare-and-exchange operation has a constant complexity, $O(1)$.

Parallel Bubble Sort

Iteration could start before previous iteration finished if does not overtake previous bubbling action:

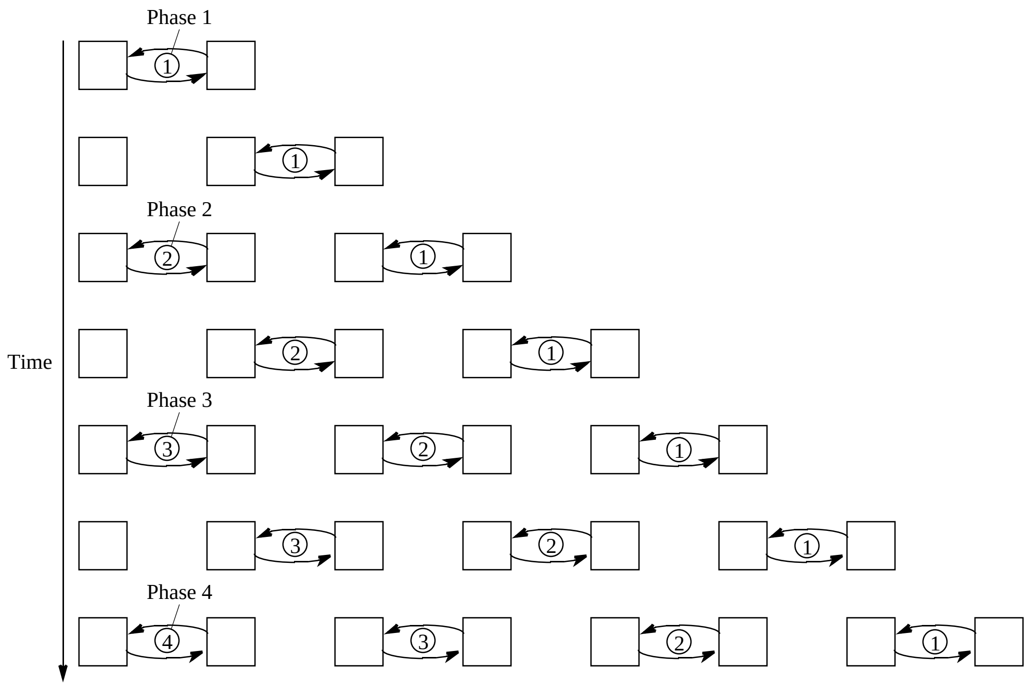


Figure 9.8 Overlapping bubble sort actions in a pipeline.

Odd-Even (Transposition) Sort

Variation of bubble sort.

Operates in two alternating phases, an *even* phase and an *odd* phase.

Even phase

Even-numbered processes exchange numbers with their right neighbor.

Odd phase

Odd-numbered processes exchange numbers with their right neighbor.

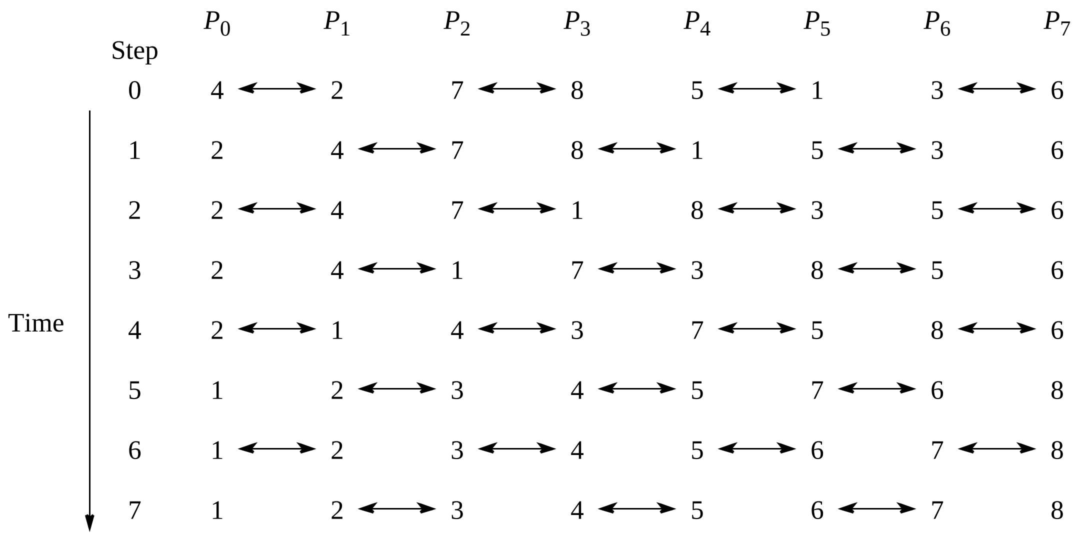


Figure 9.9 Odd-even transposition sort sorting eight numbers.

Odd-Even Transposition Sort Code

Even Phase

```
Pi, i = 0, 2, 4, …, n - 2 (even)Pi, i = 1, 3, 5, …, n - 1 (odd)

recv(&A, Pi+1);               send(&A, Pi-1);  /* even phase */
send(&B, Pi+1);               recv(&B, Pi-1);
if (A > B) B = A;             if (A > B) A = B;/* exchange */
```

where the number stored in $P_{i\ \text{(even)}}$ is B and the number stored in $P_{i\ \text{(odd)}}$ is A.

Odd Phase

```
Pi, i = 1, 3, 5, …, n - 3 (odd)Pi, i = 2, 4, 6, …, n - 2 (even)

send(&A, Pi+1);               recv(&A, Pi-1);    /* odd phase */
recv(&B, Pi+1);               send(&B, Pi-1);
if (A > B) A = B;             if (A > B) B = A; /* exchange */
```

Combined

P_i, i = 1, 3, 5, …, n - 3 (odd)

```
send(&A, P_{i-1});
recv(&B, P_{i-1});
if (A > B) A = B;
if (i <= n-3) {
    send(&A, P_{i+1});
    recv(&B, P_{i+1})
    if (A > B) A = B;
}
```

P_i, i = 0, 2, 4, …, n - 2 (even)

```
recv(&A, P_{i+1});  /* even phase */
send(&B, P_{i+1});
if (A > B) B = A;
if (i >= 2) {   /* odd phase */
  recv(&A, P_{i-1});
  send(&B, P_{i-1});
  if (A > B) B = A;
}
```

Two-Dimensional Sorting

The layout of a sorted sequence on a mesh could be row by row or *snakelike*. In a snakelike sorted list, the numbers are arranged in nondecreasing order:

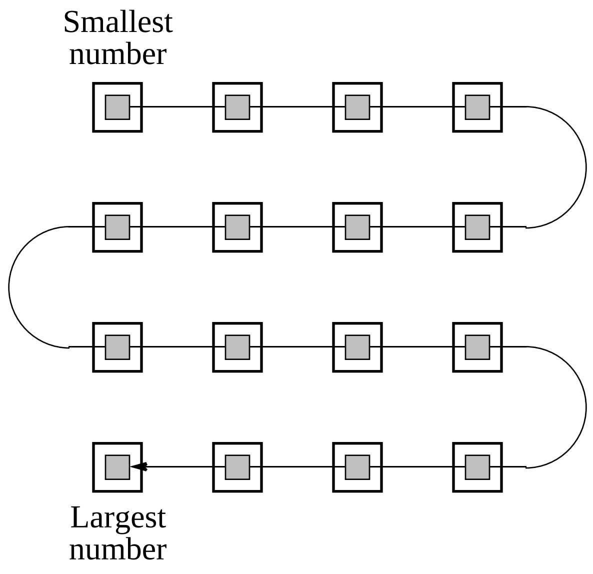


Figure 9.10 Snakelike sorted list.

Shearsort

Requires $\sqrt{n}(\log n + 1)$ steps for n numbers on a $\sqrt{n} \times \sqrt{n}$ mesh.

Odd phase

Each row of numbers is sorted independently, in alternative directions:

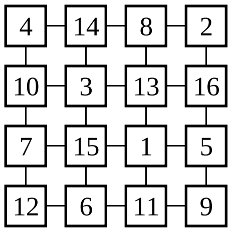
Even rows — The smallest number of each column is placed at the rightmost end and largest number at the leftmost end.
Odd rows — The smallest number of each column is placed at the leftmost end and the largest number at the rightmost end.

Even phase

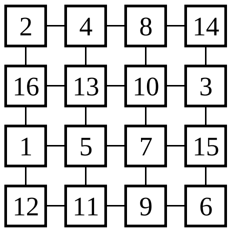
Each column of numbers is sorted independently, placing the smallest number of each column at the top and the largest number at the bottom.

After $\log n + 1$ phases, numbers sorted with a snakelike placement in mesh.

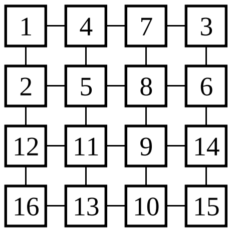
The alternating directions of the row sorting phase matches final snakelike layout.



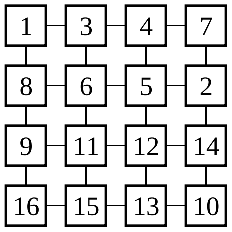
(a) Original placement of numbers



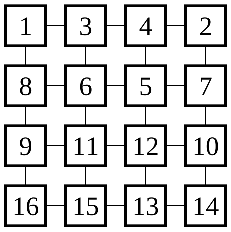
(b) Phase 1 — Row sort



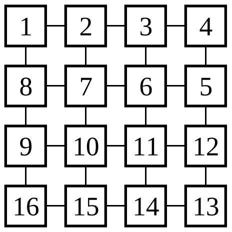
(c) Phase 2 — Column sort



(d) Phase 3 — Row sort



(e) Phase 4 — Column sort



(f) Final phase — Row sort

Figure 9.11 Shearsort.

Using Transposition

A transpose operation causes the elements in each column to be in positions in a row. Can be placed between the row operations and column operations:

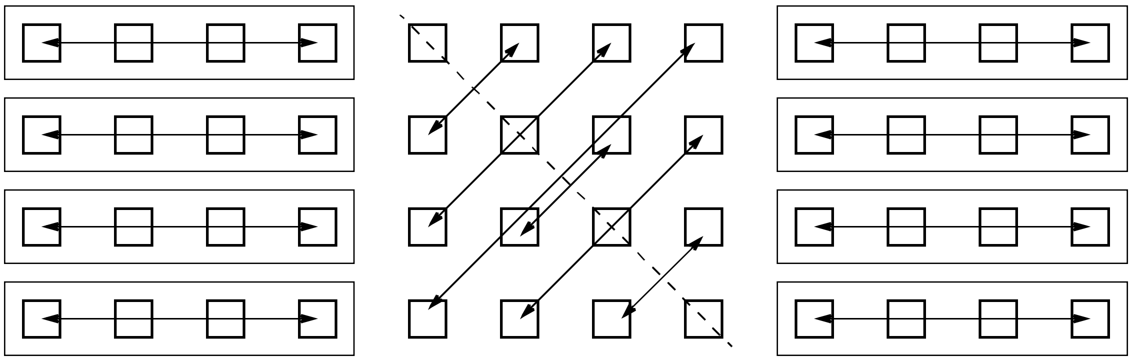

(a) Operations between elements in rows
(b) Transpose operation
(c) Operations between elements in rows (originally columns)

Figure 9.12 Using the transpose operation to maintain operations in rows.

Transposition can be achieved with $\sqrt{n}(\sqrt{n}-1)$ communications ($O(n)$).
An *all-to-all* routine could be reduce this.

Mergesort

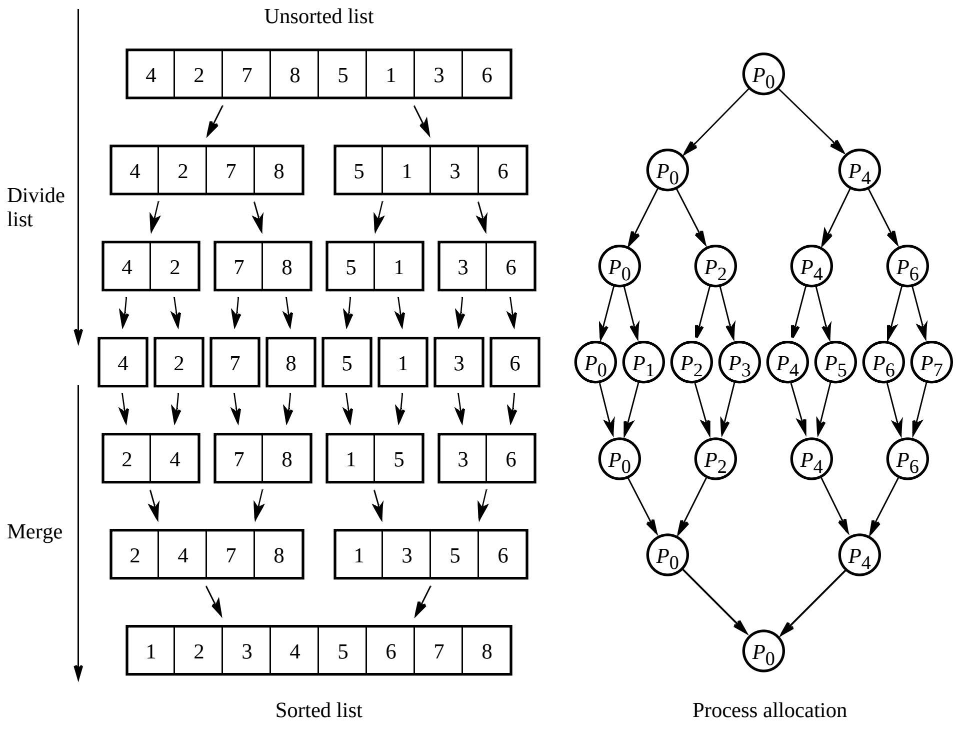


Figure 9.13 Mergesort using tree allocation of processes.

Analysis

Sequential

Sequential time complexity is $O(n \log n)$.

Parallel

There are 2 log n steps in the parallel version but each step may need to perform more than one basic operation, depending upon the number of numbers being processed.

- Next slide

Communication

In the division phase, communication only takes place as follows:

Communication at each stepProcessor communication

$t_{startup} + (n/2)t_{data}$	$P_0 \rightarrow P_4$
$t_{startup} + (n/4)t_{data}$	$P_0 \rightarrow P_2; P_4 \rightarrow P_6$
$t_{startup} + (n/8)t_{data}$	$P_0 \rightarrow P_1; P_2 \rightarrow P_3; P_4 \rightarrow P_5; P_6 \rightarrow P_7$
.	

with log p steps, given p processors. In the merge phase, the reverse communications take place:

$t_{startup} + (n/8)t_{data}$	$P_0 \rightarrow P_1; P_2 \rightarrow P_3; P_4 \rightarrow P_5; P_6 \rightarrow P_7$
$t_{startup} + (n/4)t_{data}$	$P_0 \rightarrow P_2; P_4 \rightarrow P_6$
$t_{startup} + (n/2)t_{data}$	$P_0 \rightarrow P_4$
.	

again log p steps. This leads to the communication time being

$$t_{comm} = 2(t_{startup} + (n/2)t_{data} + t_{startup} + (n/4)t_{data} + t_{startup} + (n/8)t_{data} + \dots)$$

or:

$$t_{comm} \approx 2(\log p)t_{startup} + 2nt_{data}$$

Computation

Computations only occurs in merging the sublists. Merging can be done by stepping through each list, moving the smallest found into the final list first. It takes $2n - 1$ steps in the worst case to merge two sorted lists each of n numbers into one sorted list.

Therefore, the computation consists of

$t_{comp} = 1$	$P_0; P_2; P_4; P_6$
$t_{comp} = 3$	$P_0; P_2$
$t_{comp} = 7$	P_0
.	

Hence:

$$t_{comp} = \sum_{i=1}^{\log p} (2^i - 1)$$

The parallel computational time complexity is $O(p)$ using p processors and one number in each processor.
As with all sorting algorithms, normally we would partition the list into groups, one group of numbers for each processor.

Quicksort

Sequential time complexity of $O(n\log n)$. The question to answer is whether a parallel version can achieve the time complexity of $O(\log n)$ with n processors.

Quicksort sorts a list of numbers by first dividing the list into two sublists, as in mergesort.
All the numbers in one sublist are arranged to be smaller than all the numbers in the other sublist.
Achieved by first selecting one number, called a *pivot*, against which every other number is compared.
If the number is less than the pivot, it is placed in one sublist. Otherwise, it is placed in the other sublist.
By repeating the procedure sufficiently, we are left with sublists of one number each.
With proper ordering of the sublists, a sorted list is obtained.

Sequential Code

Suppose an array **`list[]`** holds the list of numbers and **`pivot`** is the index in the array of the final position of the pivot:

```
    quicksort(list, start, end)
    {
      if (start < end) {
        partition(list, start, end, pivot)
        quicksort(list, start, pivot-1);/* recursively call on sub-
lists*/
        quicksort(list, pivot+1, end);
      }
    }
```

`Partition()` moves numbers in list between **`start`** to **`end`** so that those less than the pivot are before the pivot and those equal or greater than the pivot are after the pivot.

The pivot is in its final position of the sorted list.

Parallelizing Quicksort

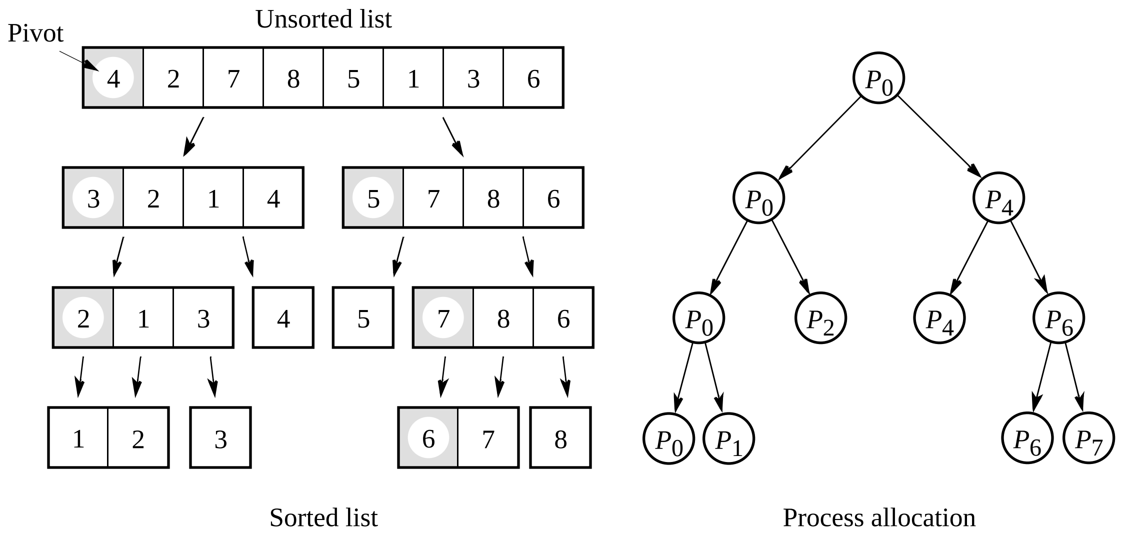


Figure 9.14 Quicksort using tree allocation of processes.

With the pivot being withheld

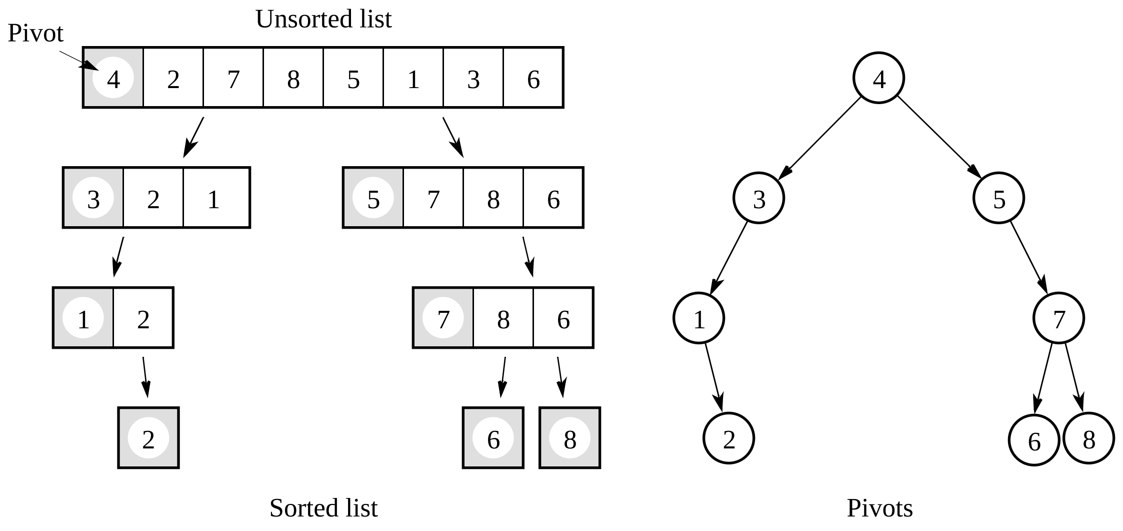


Figure 9.15 Quicksort showing pivot withheld in processes.

Analysis

Fundamental problem with all tree constructions – initial division done by a single processor, which will seriously limit speed. Suppose pivot selection is ideal and each division creates two sublists of equal size.

Computation

First, one processor operates upon n numbers. Then two processors each operate upon $n/2$ numbers. Then four processors each operate upon $n/4$ numbers, and so on:

$$t_{\text{comp}} = n + n/2 + n/4 + n/8 + \ldots \approx 2n$$

Communication

Communication also occurs in a similar fashion as for mergesort:

$$t_{\text{comm}} = (t_{\text{startup}} + (n/2)t_{\text{data}}) + (t_{\text{startup}} + (n/4)t_{\text{data}}) + (t_{\text{startup}} + (n/8)t_{\text{data}}) + \ldots$$
$$\approx (\log p)t_{\text{startup}} + nt_{\text{data}}$$

Tree in quicksort will not, in general, be perfectly balanced Pivot selection very important to make quicksort operate fast.

Work Pool Implementation

First, the work pool holds the initial unsorted list given to first processor whcih divides list into two parts. One part returned to work pool to be given to another processor, while the other part is operated upon again.

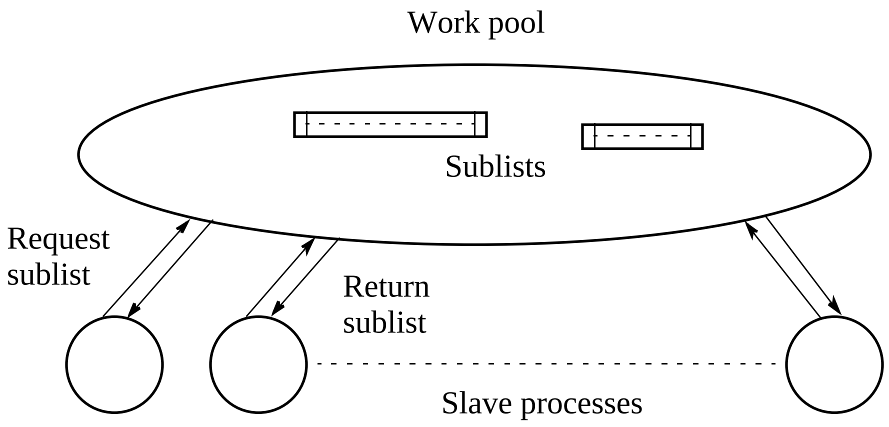


Figure 9.16 Work pool implementation of quicksort.

Quicksort on a Hypercube

Complete List Placed in One Processor

List divided into two parts using a pivot determined by processor, with one part sent to adjacent node in highest dimension. Then the two nodes can repeat the process, dividing their lists into two parts using locally selected pivots. One part is sent to a node in the next highest dimension. Continued for log *d* steps for a *d*-dimensional hypercube.

	Node		Node	
1st step:	000	→	001	(numbers greater than a pivot, say p_1)
2nd step:	000	→	010	(numbers greater than a pivot, say p_2)
	001	→	011	(numbers greater than a pivot, say p_3)
3rd step:	000	→	100	(numbers greater than a pivot, say p_4)
	001	→	101	(numbers greater than a pivot, say p_5)
	010	→	110	(numbers greater than a pivot, say p_6)
	011	→	111	(numbers greater than a pivot, say p_7)

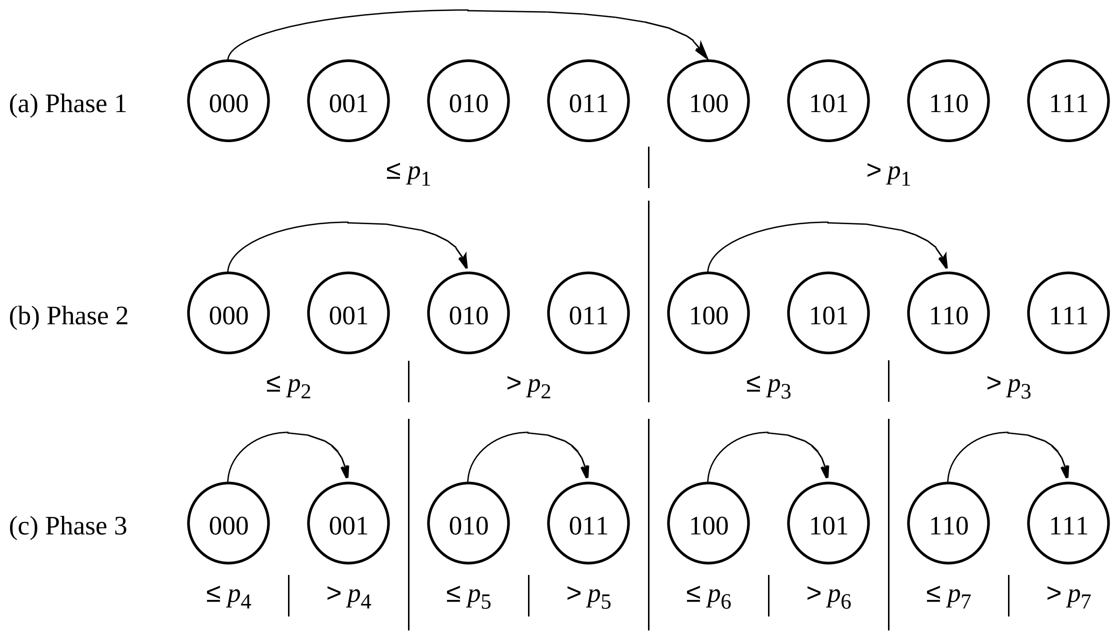


Figure 9.17 Hypercube quicksort algorithm when the numbers are originally in node 000.

Numbers Initially Distributed across All Processors

Steps

1. One processor (say P_0) selects (or computes) a suitable pivot and broadcasts this to all others in the cube.

2. Processors in “lower” subcube send their numbers which are greater than pivot to their partner processor in “upper” subcube. Processors in “upper” subcube send their numbers which are equal to or less than the pivot to their partner processor in “lower” cube.

3. Each processor concatenates the list received with what remains of its own list.

After these steps the numbers in lower subcube will all be equal to or less than the pivot and all the numbers in upper subcube will be greater than pivot. Steps 2 and 3 are now repeated recursively on the two subcubes. One process in each subcube computes a pivot for its subcube and broadcasts it throughout its subcube. These actions terminate after log d recursive phases. Suppose the hypercube has three dimensions. The numbers in processor 000 will be smaller than numbers in processor 001, which will be smaller than numbers in processor 010, and so on.

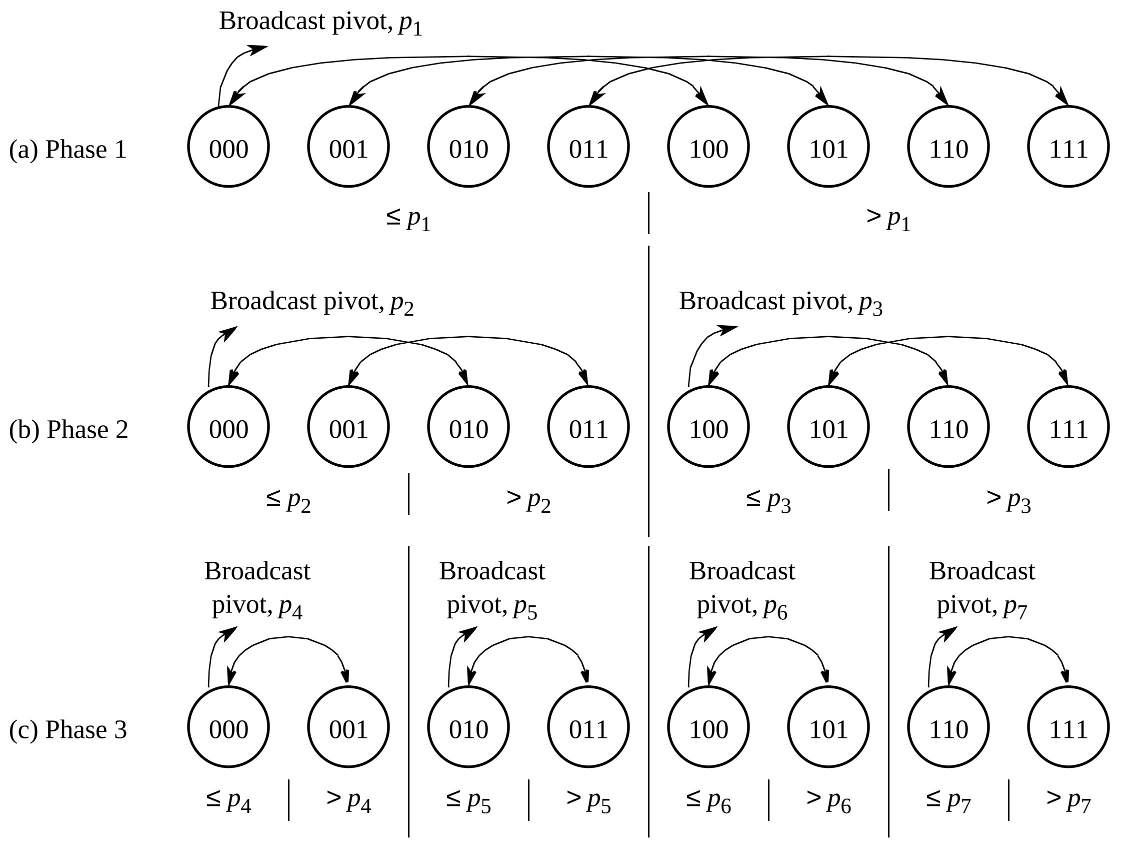


Figure 9.18 Hypercube quicksort algorithm when numbers are distributed among nodes.

Communication Patterns in Hypercube

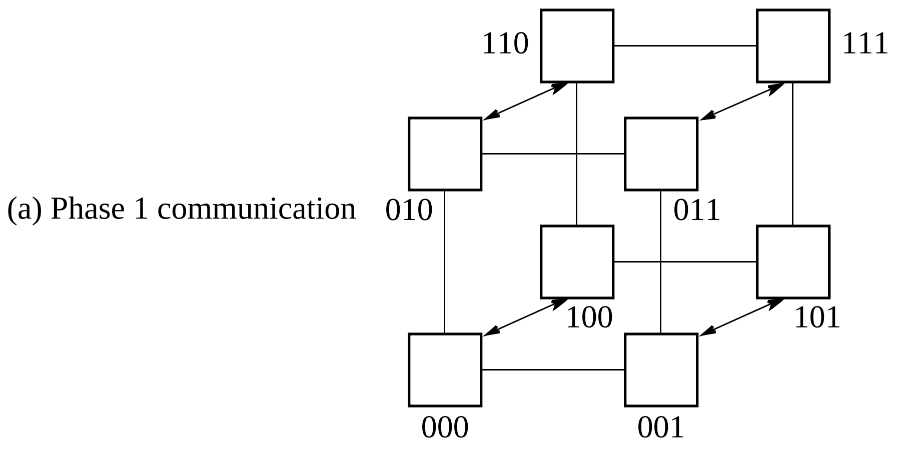
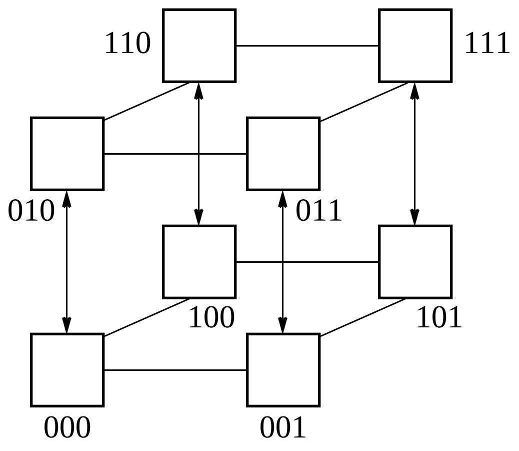
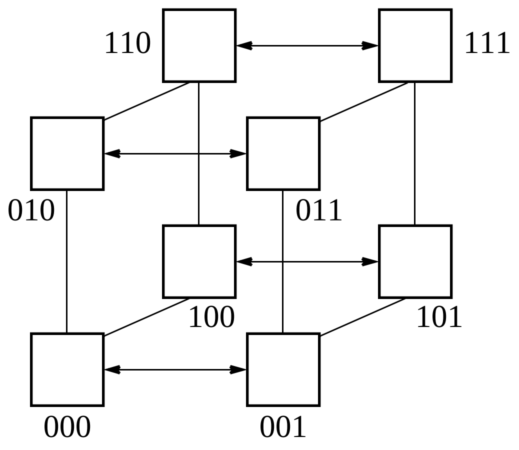


Figure 9.19 Hypercube quicksort communication.

(b) Phase 2 communication



(c) Phase 3 communication



Pivot Selection

Poor pivot selection could result in most of the numbers being allocated to a small part of the hypercube, leaving the rest idle. This is most deleterious in the first split.
In sequential quicksort algorithm, often pivot is simply chosen to be first number in the list, which could be obtained in a single step or with $O(1)$ time complexity.

One approach – take a sample of a numbers from the list, compute the mean value, and select the median as the pivot. The numbers sampled would need to be sorted at least halfway through to find the median.

We might choose a simple bubble sort, which can be terminated when the median is reached.

Hyperquicksort

Sorts numbers at each stage to maintain sorted numbers in each processor. Simplifies selecting pivots and eliminates final sorting operation.

Steps

1. Each processor sorts its list sequentially.

2. One processor (say P_0) selects (or computes) a suitable pivot and broadcasts this pivot to all others in the cube.

3. The processors in the “lower” subcube send their numbers, which are greater than the pivot, to their partner processor in the “upper” subcube. The processors in the “upper” subcube send their numbers, which are equal to or less than the pivot, to their partner processor in the “lower” cube.

4. Each processor merges the list received with its own to obtain a sorted list.

Steps 2, 3, and 4 are repeated (d phases in all for a d-dimensional hypercube).

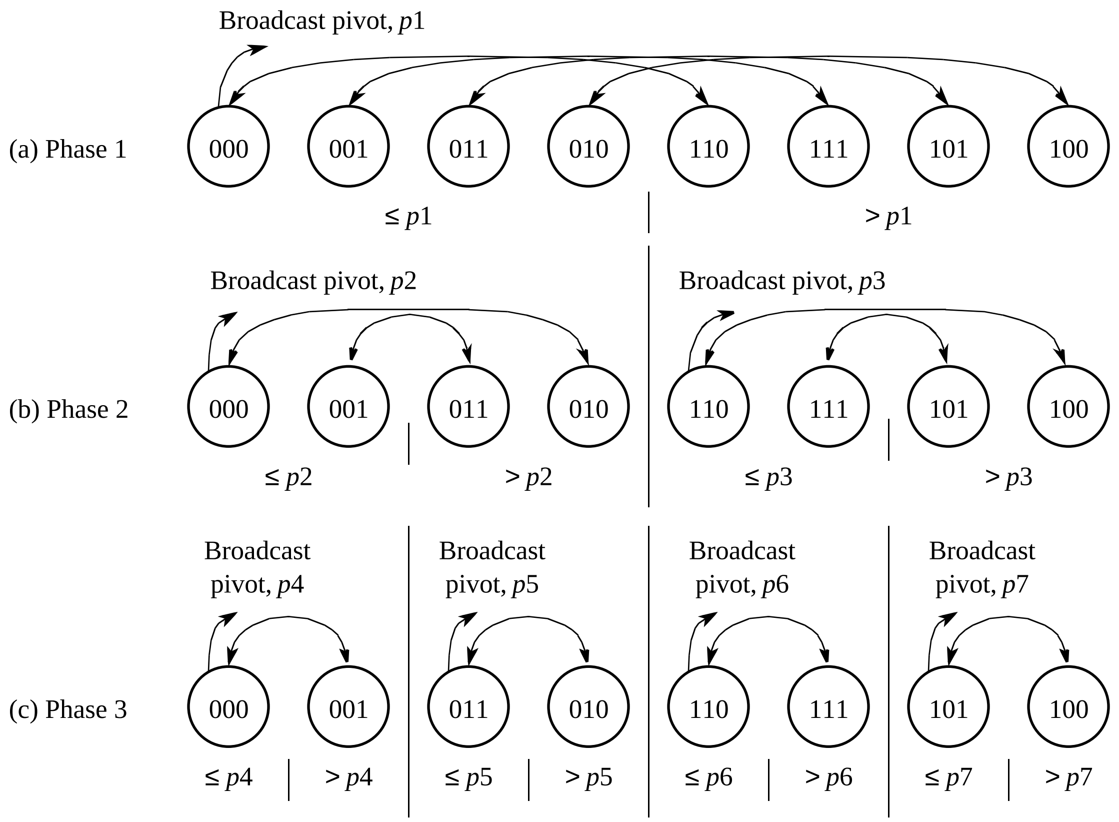


Figure 9.20 Quicksort hypercube algorithm with Gray code ordering.

Analysis

Initially, each processor has n/p numbers. Afterward, it will vary. Let it be x. Algorithm has d phases. After initial sorting step requiring $O(n/p \log n/p)$, each phase has pivot selection, pivot broadcast, a data split, data communication, and data merge.

Computation — Pivot Selection

With a sorted list, pivot selection can be done in one step, O(1), if there always were n/p numbers. In the more general case, the time complexity will be higher.

Communication — Pivot Broadcast

$$\frac{d(d-1)}{2}(t_{\text{startup}} + t_{\text{data}})$$

Computation — Data Split

If the numbers are sorted and there are x numbers, split operation done in log x steps.

Communication — Data from Split

$$t_{\text{startup}} + \frac{x}{2}t_{\text{data}}$$

Computation — Data Merge

To merge two sorted lists into one sorted list requires x steps if biggest list has x numbers.

Total - sum of the individual communication times and computation times.

Odd-Even Mergesort

Odd-Even Merge Algorithm

Will merge two *sorted* lists into one sorted list. Given two sorted lists $a_1, a_2, a_3, \ldots, a_n$ and $b_1, b_2, b_3, \ldots, b_n$ (where n is a power of 2), the following actions are performed:

1. The elements with odd indices of each sequence — that is, $a_1, a_3, a_5, \ldots, a_{n-1}$, and $b_1, b_3, b_5, \ldots, b_{n-1}$ — are merged into one sorted list, $c_1, c_2, c_3, \ldots, c_n$.
2. The elements with even indices of each sequence — that is, $a_2, a_4, a_6, \ldots, a_n$, and $b_2, b_4, b_6, \ldots, b_n$ — are merged into one sorted list, $d_1, d_2, \ldots, d_n$.
3. The final sorted list, $e_1, e_2, \ldots, e_{2n}$, is obtained by the following:

   $e_{2i} = \min\{c_{i+1}, d_i\}$

   $e_{2i+1} = \max\{c_{i+1}, d_i\}$

for $1 \le i \le n-1$. Essentially the odd and even index lists are interleaved, and pairs of odd/even elements are interchanged to move the larger toward one end, if necessary.

First number is $e_1 = c_1$ (since this is smallest of first elements of each list, a_1 or b_1) and last number is $e_{2n} = d_n$ (since this is largest of last elements of each list, a_n or b_n).

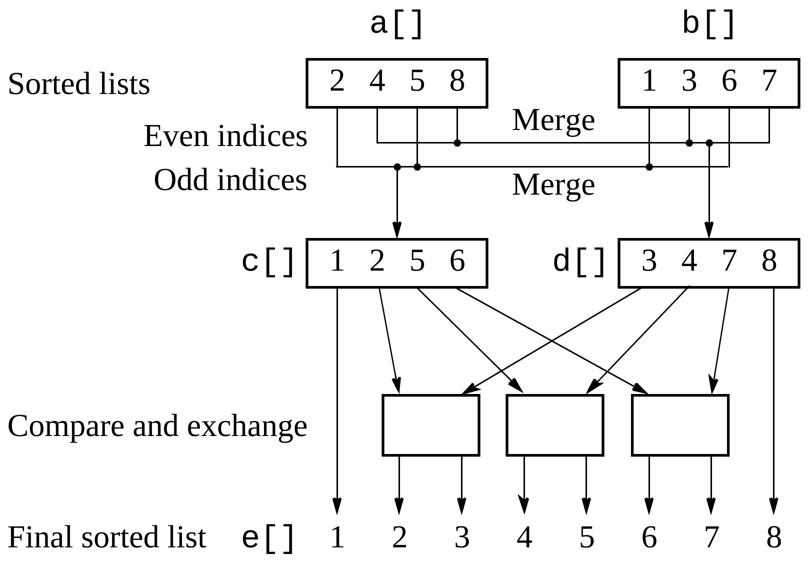


Figure 9.21 Odd-even merging of two sorted lists.

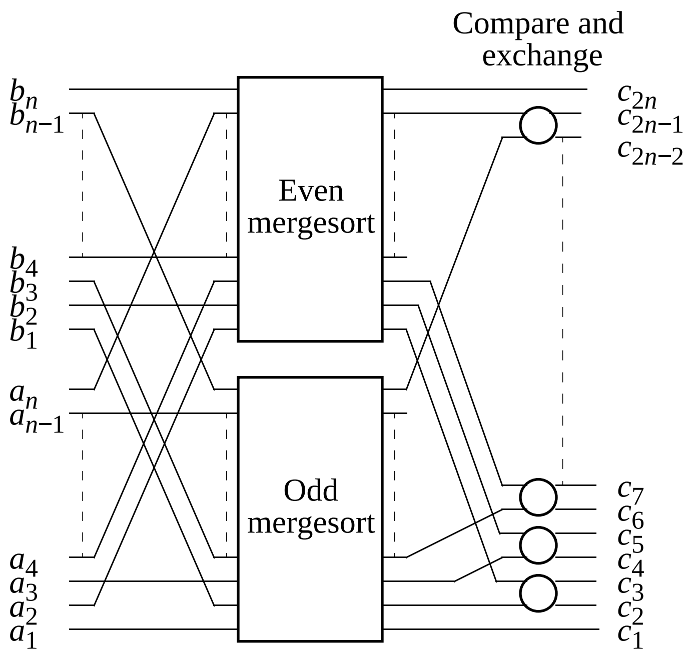


Figure 9.22 Odd-even mergesort.

Bitonic Mergesort

Bitonic Sequence

A monotonic increasing sequence is a sequence of increasing numbers.

A *bitonic sequence* has two sequences, one increasing and one decreasing. e.g.

$$a_0 < a_1 < a_2, a_3, \ldots, a_{i-1} < a_i > a_{i+1}, \ldots, a_{n-2} > a_{n-1}$$

for some value of i $(0 \le i < n)$.

A sequence is also bitonic if the preceding can be achieved by shifting the numbers cyclically (left or right).

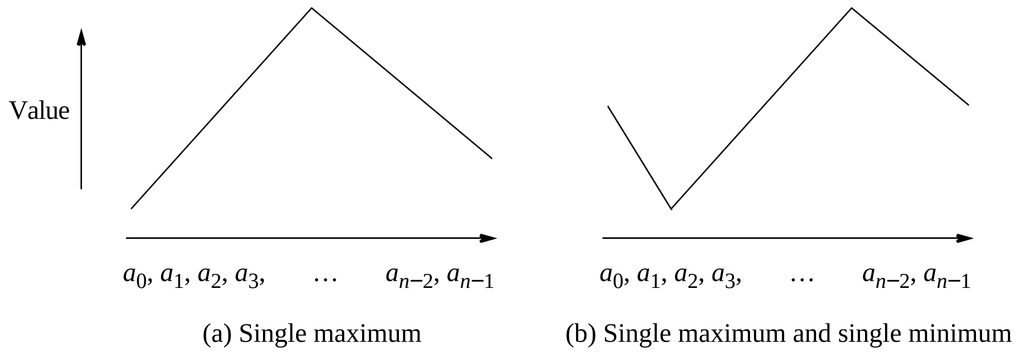

Value
$a_0, a_1, a_2, a_3, \dots a_{n-2}, a_{n-1}$
(a) Single maximum
$a_0, a_1, a_2, a_3, \dots a_{n-2}, a_{n-1}$
(b) Single maximum and single minimum

Figure 9.23 Bitonic sequences.

“Special” Characteristic of Bitonic Sequences

If we perform a compare-and-exchange operation on a_i with $a_{i+n/2}$ for all i , where there are n numbers in the sequence, get two bitonic sequences, where the numbers in one sequence are all less than the numbers in the other sequence.

Example

Starting with the bitonic sequence

3, 5, 8, 9, 7, 4, 2, 1

we get the sequences shown below

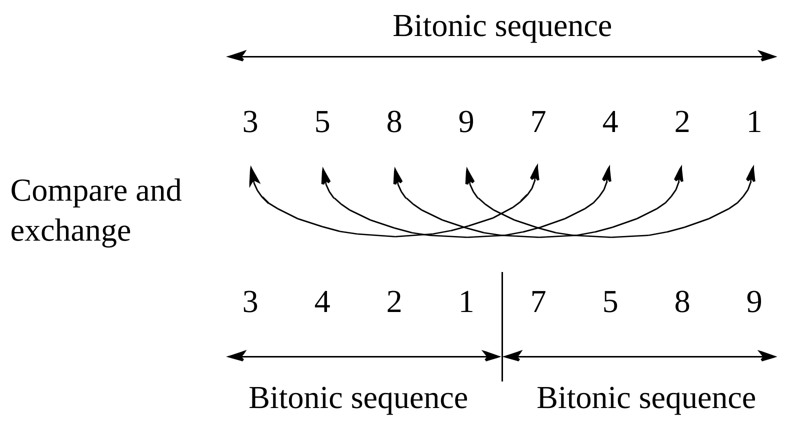


Figure 9.24 Creating two bitonic sequences from one bitonic sequence.

Compare-and-exchange operation moves smaller numbers of each pair to the left sequence and larger numbers of the pair to the right sequence. Given a bitonic sequence, recursively performing compare-and-exchange operations will sort the list.

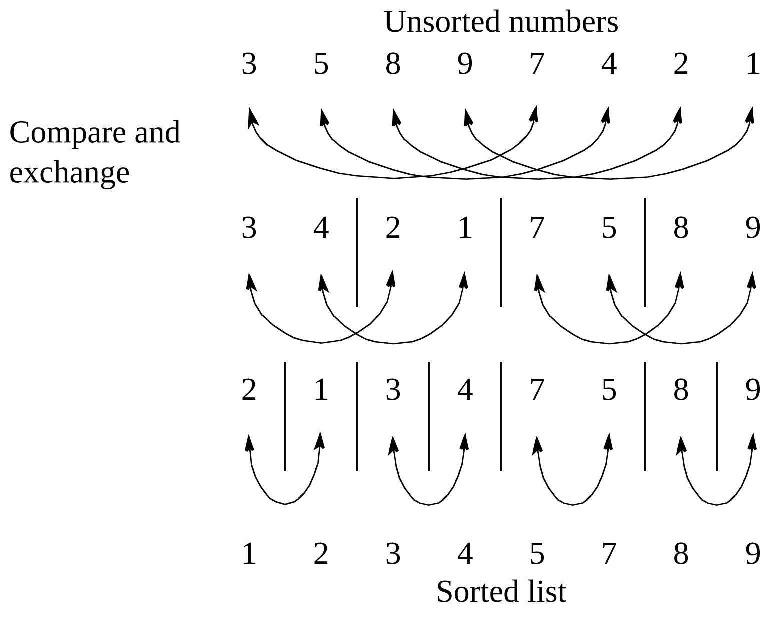


Figure 9.25 Sorting a bitonic sequence.

Sorting

To sort an unordered sequence, sequences are merged into larger bitonic sequences, starting with pairs of adjacent numbers.

By a compare-and-exchange operation, pairs of adjacent numbers are formed into increasing sequences and decreasing sequences, pairs of which form a bitonic sequence of twice the size of each of the original sequences.

By repeating this process, bitonic sequences of larger and larger lengths are obtained.

In the final step, a single bitonic sequence is sorted into a single increasing sequence.

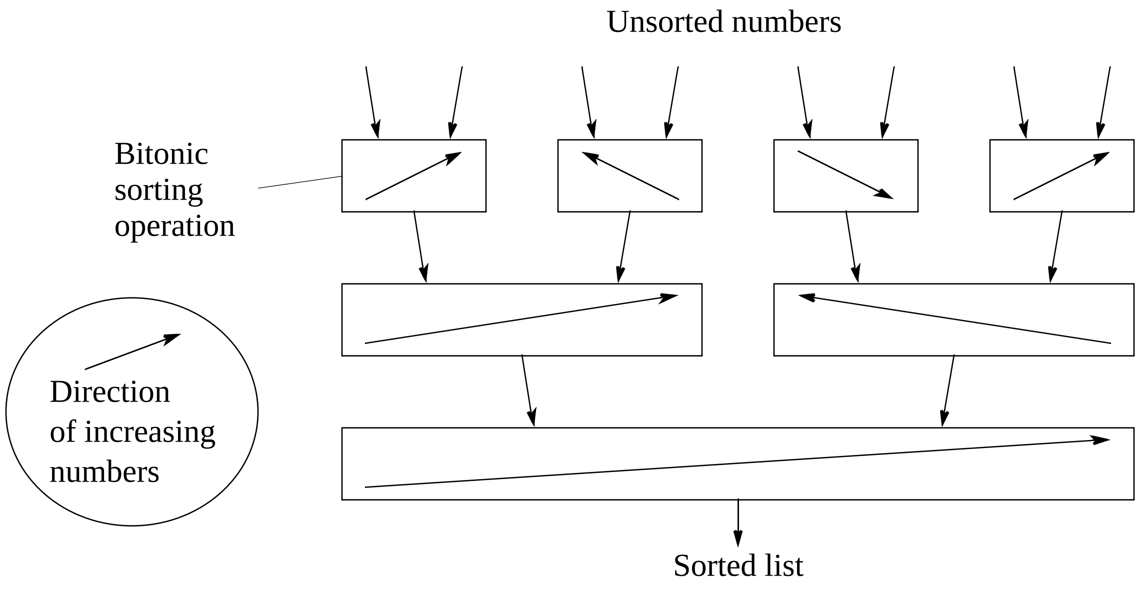
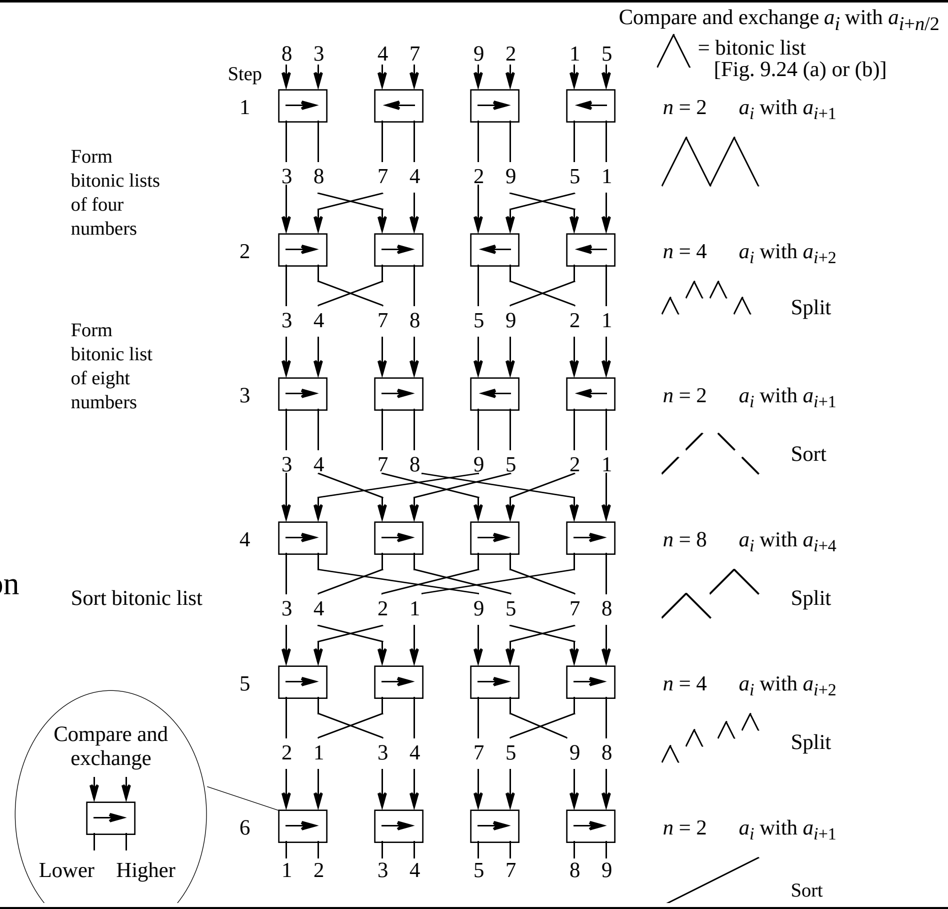


Figure 9.26 Bitonic mergesort.



Compare and exchange a_i with $a_{i+n/2}$
= bitonic list [Fig. 9.24 (a) or (b)]
Step
8 3 4 7 9 2 1 5
1
$n = 2$ a_i with a_{i+1}
Form bitonic lists of four numbers
3 8 7 4 2 9 5 1
2
$n = 4$ a_i with a_{i+2}
Split
3 4 7 8 5 9 2 1
Form bitonic list of eight numbers
3
$n = 2$ a_i with a_{i+1}
Sort
3 4 7 8 9 5 2 1
4
$n = 8$ a_i with a_{i+4}
Sort bitonic list
Split
3 4 2 1 9 5 7 8
5
$n = 4$ a_i with a_{i+2}
Split
2 1 3 4 7 5 9 8
Compare and exchange
6
$n = 2$ a_i with a_{i+1}
Lower Higher
1 2 3 4 5 7 8 9
Sort

Figure 9.27 Bitonic mergesort on eight numbers.

Phases

The six steps (for eight numbers) are divided into three phases:

Phase 1 (Step 1) Convert pairs of numbers into increasing/decreasing sequences and hence into 4-bit bitonic sequences.

Phase 2 (Steps 2/3) Split each 4-bit bitonic sequence into two 2-bit bitonic sequences, higher sequences at center.

Sort each 4-bit bitonic sequence increasing/decreasing sequences and merge into 8-bit bitonic sequence.

Phase 3 (Steps 4/5/6) Sort 8-bit bitonic sequence (as in Figure 9.27).

Number of Steps

In general, with $n = 2^k$, there are k phases, each of 1, 2, 3, …, k steps. Hence the total number of steps is given by

$$\text{Steps} = \sum_{i=1}^{k} i = \frac{k(k+1)}{2} = \frac{\log n(\log n + 1)}{2} = O(\log^2 n)$$