An Analysis of Propogation in Graphs Modulo a Prime

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1 Propogation in Graphs

Let X be a graph, and let $A = A(X)$ be its adjacency matrix. Let $V(X)$ be the vertices and $E(X)$ be the edges. By a *state* s of X we mean an assignment of a number to each vertex. We define an operation f on the numberings as follows:

$$(f(s))(v) = \sum_{w \sim v} s(w) \tag{1}$$

If we think of s as a vector (in R^X for some ring R), then this is

$$f(s) = As \tag{2}$$

We call the operation of applying f to a given state *propogating* the graph. We are interested in studying what happens mod n if the state s assigns an integer to each vertex and we repeatedly propogate the graph.

2 Terminating States

The first question we consider is which states will eventually become the state with all 0s upon repeated propogation. We ask whether all states will go to zero in this case. Note that this means that $A^k s = 0$ for all s for some k. Since we are working mod n, there are only finitely many possible A, so such a k is finite. Thus in particular $A^k e_i = 0$, where e_i is the vector with ith coordinate 1 and all other coordinates 0. So $A^k I = 0$, where I is the identity matrix. But this means that $A^k = 0$, so this occurs if and only if A is nilpotent. Also note that if a state terminates mod p and mod q, then it terminates mod pq, as follows: After some number of propogations, all entries are divisible by p. Now factor out p from each entry. After some number of propogations, the resulting state will have all entries divisible by q. Putting back in the factor of p yields that all entries will be divisible by pq. Note that this reasoning does not require that p and q be relatively prime. Thus it suffices to consider when a state terminates modulo a given prime p. We have the following theorem:

Theorem 2.1 *If A is a linear operator on a finite-dimensional vector space V (over an arbitrary field $\mathbb{F}$), then A is nilpotent iff all of its eigenvalues are zero.*

This will usually be the most convenient way to characterize terminating operators.

2.1 k-dimensional grids

We first consider a graph consisting of all points $(x_1, \ldots, x_k)$ with $0 \leq x_i \leq d_i$, where two points are connected if all but one of the coordinates are the same, and the final coordinate differs by 1. We call this a $d_1 \times d_2 \times \cdots \times d_k$ grid. Note that, as a graph, this is the same as $P_{d_1}\square P_{d_2}\square\cdots\square P_{d_k}$, where P_m is the path of length m and $\square$ is the Cartesian product of graphs. We have the following characterization:

Theorem 2.2 *The $d_1\times\cdots\times d_k$ grid is nilpotent mod p if and only if one of the following hold:*

1. *$p=2$,and each d_i is either one less than a power of 2 or 2. Furthermore, the number of indices i with $d_i=2$ is even.*
2. *$d_i=1$ for all i.*

Proof Clearly the graph is nilpotent if all d_i are equal to 1, so we assume throughout the rest of this proof that this is not the case.

Note that if $f_1,\ldots,f_k$ are eigenfunctions of $P_{d_1},\ldots,P_{d_k}$ with eigenvalues $\lambda_1,\ldots,\lambda_k$, then $(f_1\otimes\cdots\otimes f_k)(x_1,\ldots,x_k) := f_1(x_1)\cdots f_k(x_k)$ is an eigenfunction of $P_{d_1}\square\cdots\square P_{d_k}$ with eigenvalue $\lambda_1+\ldots+\lambda_k$. In fact, $A(P_{d_1}\square\cdots\square P_{d_k}) = A(P_{d_1})\otimes I_{d_2}\otimes\cdots\otimes I_{d_k}+\ldots+I_{d_1}\otimes\cdots\otimes I_{d_{k-1}}\otimes A(P_{d_k})$, so this fully characterizes the eigenfunctions of $P_{d_1}\square\cdots\square P_{d_k}$. Thus it suffices to consider the spectrum of P_m mod p.

Our motivation will come from the spectrum of P_m over $\mathbb{C}$, where the eigenfunctions are $f_a(v) = e^{\frac{2\pi i a v}{m+1}} - e^{\frac{-2\pi a v}{m+1}}$ with eigenvalues $\lambda_a = e^{\frac{2\pi i a}{m+1}} + e^{\frac{-2\pi i a}{m+1}}$. We can find an analog mod p by considering an $(m+1)$st root of unity ζ and considering

$$f_a(v) = \zeta^{av} - \zeta^{-av}$$

Then it is easy to see that this is an eigenfunction with eigenvalue $\lambda_a = \zeta^a + \zeta^{-a}$. Now let $m+1 = p^x k$, where k is not divisible by p. Then consider $\mathbb{F}_{p^{\phi(k)}}$. This field's multiplicative group has order $p^{\phi(k)}-1$, so that by Euler's theorem it has order divisible by k, whence it has an element of order k. Let ζ be this element and consider the eigenfunction above. First, we check that $f_a(v)$ is not the zero function. Suppose that $\zeta^v - \zeta^{-v} = 0$. Then $\zeta^{2v}=1$, so $(\zeta^2)^v = 1$ for all v. This implies that $\zeta^2 = 1$ for $m>1$. Thus, unless $k=1,2$ (corresponding to $m=\{1,2\}p^x-1$), we have a non-zero eigenfunction. In these cases, we note that $f(v)=v$ is an eigenfunction with eigenvalue 2, so that unless $p=2$ we have a non-zero eigenfunction with non-zero eigenvalue. Next, we check that $\lambda_a\neq 0$ for some a. Suppose that $\lambda_1=0$. Then $\zeta+\zeta^{-1}=0$. Thus $\zeta^2+1=0$. Now suppose that $\lambda_2=0$. Then $\zeta^2+\zeta^{-2}=0$, so $\zeta^4+1=0$. But $\zeta^2=-1$, so $\zeta^4=(\zeta^2)^2=(-1)^2=1$. Thus, since $\zeta^4+1=0$, $1+1=0$ so $p=2$. On the other hand, if $\zeta^2+1=0$, then ζ has order at most 2 over $\mathbb{F}_2$, so $\zeta\in\mathbb{F}_4$. But every non-zero element in $\mathbb{F}_4$ has order 1 or 3. Thus $m+1\mid 3$ so $m=2$. We will analyze this case in Section **??**. We have thus shown that for $p\neq 2$, each path has at least one non-zero eigenvalue. Since all of these graphs are bipartite, this actually implies that each path has at least two distinct eigenvalues. Therefore, any Cartesian product of the paths will also have at least one non-zero eigenvalue, since the eigenvalues are the sum of the eigenvalues of each of the paths, and so for the last path in the product we can choose between two different eigenvalues in the sum, so that both sums cannot be zero. We thus can confine our attention to the case when $p=2$.

In fact, the only places where we need to do extra analysis are for $m = 2$ and $m = 2^x - 1$, as these are the only places in the preceding analysis where we needed to assume $p \neq 2$. We show that the only eigenvalue of $A(P_2)$ over $\mathbb{F}_2$ is 1 in Section **??**. If $m = 2^x - 1$, we have the following argument to show that all states terminate: We proceed by induction on i, saying that all states in P_{2^i-1} terminate. For $i = 1$ the result is trivial. Otherwise, let s be a state in $P_{2^{i+1}-1}$. We define a function $g : \mathbb{F}_2^{P_{2^{i+1}-1}} \to \mathbb{F}_2^{P_{2^i-1}}$ as $(g(s))(v) = s(v) + s(2^{i+1} - v)$, where we abuse notation and associate the first $2^i - 1$ vertices in $P_{2^{i+1}-1}$ with the vertices in P_{2^i-1}. It is easy to verify that $A(P_{2^i-1})g = gA(P_{2^{i+1}-1})$. Also, g is clearly surjective. Thus since $A(P_{2^i-1})$ is nilpotent by the inductive hypothesis, $A(P_{2^{i+1}_1})$ must be nilpotent as well, which completes our induction.

We also need to show that, in all other cases, there are at least two distinct eigenvalues (we can not use the bipartite condition anymore since $x = -x$ in $\mathbb{F}_2$). This argument goes much the same as before. Suppose that $\lambda_1 = \lambda_2$. Then $\zeta + \zeta^{-1} = \zeta^2 + \zeta^{-2}$. Thus $\zeta^3 + \zeta = \zeta^4 + 1$, which over $\mathbb{F}_2$ becomes $\zeta(\zeta + 1)^2 = (\zeta + 1)^4$, or $(\zeta + 1)^2(\zeta^2 + \zeta + 1) = 0$. Thus either $\zeta = 1$ or ζ has degree 2 over $\mathbb{F}_2$. We have already shown that we can choose $\zeta \neq 1$ when $m \neq 2^x - 1$. Thus ζ has degree 2 over $\mathbb{F}_2$, so ζ has order 1 or 3, so $m + 1 \mid 3$ and $m = 2$.

If any $d_i \neq 2, 2^x - 1$ in our Cartesian product, then we can use the same argument as above to show that we can find two distinct sums of eigenvalues (since at least one path has at least two distinct eigenvalues), and not both sums can be zero. Thus if our graph is nilpotent, each $d_i = 2, 2^x - 1$. If there are an odd number of d_i, then all eigenvalues of the resulting graph are 1 (so that *all* states are fixed by propogation), so again the graph is not nilpotent. On the other hand, if there are an even number of d_i, then all eigenvalues are 0, so the graph is nilpotent, as our characterization requires. We have thus completed the desired characterization.

We now devote our attention to the case of $A(P_2)$ over $\mathbb{F}_2$.

2.1.1 Spectrum of $A(P_2)$ over $\mathbb{F}_2$

It is easy to explicitly calculate the characteristic polynomial in this case. We see that $A(P_2)$ is equal to

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3}$$

This has characteristic polynomial $\lambda^2 - 1 = (\lambda - 1)^2$ over $\mathbb{F}_2$, from which we see that both eigenvalues are 1.