The Measurable Chromatic Number of the Odd-Distance Graph is Infinite

Jacob Steinhardt

Let $\mathcal{O}$ be the graph with $V(\mathcal{O}) = \mathbb{R}^2$ and where two vertices are connected if they are at an odd distance from each other. We call $\mathcal{O}$ the *odd-distance graph*. We aim to show that the chromatic number χ of $\mathcal{O}$ is infinite if we only allow measurable colorings (from now on, by chromatic number we mean chromatic number in this sense). Consider the operator $B_\alpha : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ defined by

$$(B_\alpha f)(x,y) = \int_{-\pi}^{\pi} \sum_{k=0}^{\infty} \alpha^{-k} f(x + (2k+1)\cos(\theta), y + (2k+1)\sin(\theta))d\theta \tag{1}$$

Clearly, B_α is a linear operator. We also make the following observation:

Lemma 0.1 *Let I be an independent set in $\mathcal{O}$, and let g be any function that is zero outside of I. Then $\langle f, B_\alpha f\rangle = 0$.*

Proof

$$\begin{aligned}
\langle f, B_\alpha f\rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)(B_\alpha f)(x,y)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\int_{-\pi}^{\pi}\sum_{k=0}^{\infty}\alpha^{-k} f(x+(2k+1)\cos(\theta), y+(2k+1)\sin(\theta))d\theta dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\pi}^{\pi}\sum_{k=0}^{\infty}\alpha^{-i} f(x,y)f(x+(2k+1)\cos(\theta), y+(2k+1)\sin(\theta))d\theta dxdy \\
&= 0
\end{aligned}$$

In the last equality we used the fact that $f(x,y)f(x+(2k+1)\cos(\theta), y+(2k+1)\sin(\theta) = 0$ since not both (x,y) and $(x+(2k+1)\cos(\theta), y+(2k+1)\sin(\theta)$ can be in I (they are at odd distance), so at least one of the two must be zero.

We can use this to bound the chromatic number χ of $\mathcal{O}$. Let Λ be the set of eigenvalues of B. Let $\lambda_{\max} = \sup(\Lambda)$ and $\lambda_{\min} = \inf(\Lambda)$. Then we have the following:

Lemma 0.2

$$\chi \geq 1 - \frac{\lambda_{\max}}{\lambda_{\min}} \tag{2}$$

Proof Let f be any eigenfunction with eigenvalue λ. Suppose that there exists a χ-coloring of $\mathcal{O}$ with color classes $I_1, \ldots, I_\chi$. Let f_i be defined as

$$f_i(x) = \left\{ \begin{array}{ll} f(x) & x \in I_i \\ 0 & x \notin I_i \end{array} \right\} \tag{3}$$

We note that each f_i satisfies the conditions of Lemma **??**. Combining this with the fact that B_α is symmetric and that $\lambda_{\min}||f||^2 \leq \langle f, Bf \rangle$ for all f, we have (here we use the condition that the coloring classes be measurable):

$$\begin{aligned}
2(\chi - 1)\lambda_{\min}||f||^2 &= \sum_{i,j=1}^{\chi} \lambda_{\min}||f_i - f_j||^2 \\
&\leq \sum_{i,j=1}^{\chi} \langle f_i - f_j, B(f_i - f_j) \rangle \\
&= \sum_{i,j=1}^{\chi} \langle f_i, Bf_i \rangle + \langle f_j, Bf_j \rangle - 2\langle f_i, Bf_j \rangle \\
&= -2\sum_{i,j=1}^{\chi} \langle f_i, Bf_j \rangle \\
&= -2\langle \sum_{i=1}^{\chi} f_i, B(\sum_{i=1}^{\chi} f_i \rangle \\
&= -2\langle f, Bf \rangle \\
&= -2\lambda||f||^2
\end{aligned}$$

So $2(\chi - 1)\lambda_{\min}||f||^2 \leq -2\lambda||f||^2$. Re-arranging and using the fact that $\lambda_{\min} \leq 0$, we have $\chi \geq 1 - \frac{\lambda}{\lambda_{\min}}$. Letting λ approach $\lambda_{\max}$ yields the desired result.

We next compute the eigenvalues of B_α. It is easy to verify that $f_{(r,s)}(x, y) = e^{i(rx+sy)}$ is an eigenfunction of B_α. We have, from the theory of harmonic analysis on $\mathbb{R}^2$, that these functions span the set of equivalence classes of $L^2(\mathbb{R}^2)$ under the Haar measure. Thus, in particular, all eigenvalues are accounted for by these eigenfunctions. We see that the eigenvalue of the eigenfunction $f_{(r,s)}$ is given by

$$\lambda_{(r,s)} = \int_{-\pi}^{\pi} \sum_{k=0}^{\infty} \alpha^{-k} e^{i(2k+1)(r\cos(\theta)+s\sin(\theta))} d\theta = \int_{-\pi}^{\pi} \sum_{k=0}^{\infty} \alpha^{-k} e^{i(2k+1)\sqrt{r^2+s^2}\cos(\theta+\phi)} d\theta \tag{4}$$

for an appropriately chosen ϕ. Thus we need only actually consider $\lambda_{(r,0)}$, which we from now on denote $\lambda(r)$. Then we have

$$\lambda(r) = \int_{-\pi}^{\pi} \sum_{k=0}^{\infty} \alpha^{-k} \left(e^{ir\cos(\theta)}\right)^{2k+1} = \int_{-\pi}^{\pi} \frac{e^{ir\cos(\theta)}}{1 - \alpha^{-1}e^{2ir\cos(\theta)}} d\theta \tag{5}$$

Here we have simply summed the geometric series. Since B_α is symmetric, $\lambda(r)$ must be real, so we can take the real part of the integral:

$$
\begin{aligned}
\lambda(r) &= Re\left[\int_{-\pi}^{\pi} \frac{(\cos(r\cos(\theta)) + i\sin(r\cos(\theta)))(1-\alpha^{-1}\cos(2r\cos(\theta)) + i\alpha^{-1}\sin(2r\cos(\theta)))}{(1-\alpha^{-1}\cos(2r\cos(\theta)))^2 + \alpha^{-2}\sin(2r\cos(\theta))^2} d\theta\right] \\
&= \int_{-\pi}^{\pi} \frac{\cos(r\cos(\theta))(1-\alpha^{-1}\cos(2r\cos(\theta))) - \alpha^{-1}\sin(r\cos(\theta))\sin(2r\cos(\theta))}{1+\alpha^{-2} - 2\alpha^{-1}\cos(2r\cos(\theta))} d\theta \\
&= \int_{-\pi}^{\pi} \alpha\frac{\alpha\cos(r\cos(\theta)) - \cos(r\cos(\theta))\cos(2r\cos(\theta)) - \sin(r\cos(\theta))\sin(2r\cos(\theta))}{\alpha^2 + 1 - 2\alpha\cos(2r\cos(\theta))} d\theta \\
&= \int_{-\pi}^{\pi} \alpha\frac{\alpha\cos(r\cos(\theta)) - \cos(r\cos(\theta))}{\alpha^2 + 1 - 2\alpha\cos(2r\cos(\theta))} d\theta \\
&= \int_{-\pi}^{\pi} \frac{\alpha(\alpha-1)\cos(r\cos(\theta))}{(\alpha-1)^2 + 4\alpha\sin^2(r\cos(\theta))} d\theta
\end{aligned}
$$

In the second-to-last step, we used the identity $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$. It is easy to see based on Equation **??** that the maximum occurs when $r = 0$, when we get $\lambda(0) = \frac{2\pi\alpha}{\alpha-1}$. We will, on the other hand, show that the magnitude of λ_{min} is at most $O((\alpha-1)^{-\frac{3}{4}})$. This shows that as α approaches 1, $1 - \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}}$ grows without bound, so that there cannot exist any finite coloring of $\mathcal{O}$.

Note that for $r \leq \frac{\pi}{2}$, $\lambda(r)$ is necessarily positive since the integrand is always positive ($\cos(r\cos(\theta))$ being the only thing that can go negative in the expression). We thus assume that $r > \frac{\pi}{2}$. It suffices to show that

$$
\int_0^{\frac{\pi}{2}} \frac{(\alpha-1)\cos(r\cos(\theta))}{(\alpha-1)^2 + 4\alpha\sin^2(r\cos(\theta))} d\theta \leq c(\alpha-1)^{-\frac{3}{4}} + d \tag{6}
$$

for all r for some constants c, d (as this, neglecting a factor of 4α, is clearly an upper bound for the integral above). Let h be the function we are integrating. Let $\mathcal{R}_k$ denote the region for which $h(\theta) \geq 1$ and that contains the value of θ where $\cos(\theta) = \frac{k\pi}{r}$. Then we note that $|\int_{\mathcal{R}_k} h(x)dx| > |\int_{\mathcal{R}_{k-1}} h(x)dx|$ since $\cos(\theta)$ decreases faster as θ increases from 0 to $\frac{\pi}{2}$. We will bound the area of $\mathcal{R}_{\lfloor \frac{r}{\pi} \rfloor}$. First, we determine when

$$
\frac{\alpha-1}{(\alpha-1)^2 + 4\alpha\sin^2(r\cos(\theta))} \geq 1 \tag{7}
$$

as this is clearly a superset of the area where $h(\theta) \geq 1$. But this happens when $\alpha - 1 \geq (\alpha-1)^2 + 4\alpha\sin^2(r\cos(\theta))$, or $\sin^2(r\cos(\theta)) \leq \frac{(\alpha-1)-(\alpha-1)^2}{4\alpha} = (\alpha-1)\frac{2-\alpha}{4\alpha} < \frac{\alpha-1}{4}$. So the area for which (**??**) holds is contained in the area for which $\sin(r\cos(\alpha)) \in [-\frac{\sqrt{\alpha-1}}{2}, \frac{\sqrt{\alpha-1}}{2}]$. On the other hand, this is contained in the area in which $r\cos(\theta)$ is within $\sqrt{\frac{\alpha-1}{2}}$ of a multiple of π, as $\sin(\sqrt{\frac{\alpha-1}{2}}) > \sqrt{\frac{\alpha-1}{2}} - \frac{(\alpha-1)^{1.5}}{12\sqrt{2}} > \frac{\sqrt{\alpha-1}}{2}$ for $\alpha - 1$ small enough. So we want to find when

$$
-\frac{1}{r}\sqrt{\frac{\alpha-1}{2}} \leq \frac{k\pi}{r} - \cos(\theta) \leq \frac{1}{r}\sqrt{\frac{\alpha-1}{2}} \tag{8}
$$

We claim that, if $\cos(\theta_0) = \frac{k\pi}{r}$, then it suffices to take $\theta \in [\theta_0 - \frac{2\sqrt[4]{\alpha-1}}{\sqrt{r}}, \theta_0 + \frac{2\sqrt[4]{\alpha-1}}{\sqrt{r}}]$. First of all, if $\theta_0 - \frac{\sqrt{\alpha-1}}{r} < 0$ or $\theta_0 + \frac{\sqrt{\alpha-1}r}{>}\frac{\pi}{2}$, then θ is outside of our range of integration and so we are definitely covering at least the area we need on that end of the interval. Thus we may assume otherwise, and we have the following lemma:

Lemma 0.3 *If $d > 0$ and $\theta, \theta + d \in [0, \frac{\pi}{2}]$, then* $\cos(\theta) - \cos(\theta + d) \geq 1 - \cos(d)$.

Proof Take $\frac{d}{d\theta}[\cos(\theta) - \cos(\theta + d)] = \sin(\theta + d) - \sin(\theta)$. This is clearly increasing for $\theta \in [0, \frac{\pi}{2} - d]$, so we might as well take $\theta = 0$, as this gives a smaller value for $\cos(\theta) - \cos(\theta + d)$ than any legal value of θ. Then we get $1 - \cos(d)$ as our answer, as claimed.

With Lemma **??** in hand, we need only show that $1 - \cos(\frac{2\sqrt[4]{\alpha-1}}{\sqrt{r}} > \frac{1}{r}\sqrt{\frac{\alpha-1}{2}}$. This is evident once again from the Taylor approximation as, as for $\alpha - 1$ small enough, $1 - \cos(\frac{2\sqrt[4]{\alpha-1}}{\sqrt{r}} > \frac{2\sqrt{\alpha-1}}{r} - \frac{2(\alpha-1)}{3r^2} > \frac{1}{r}\sqrt{\frac{\alpha-1}{2}}$. Thus for any given value of k, the area for which (**??**) holds is at most $\frac{4\sqrt[4]{\alpha-1}}{\sqrt{r}}$. We only care about $\mathcal{R}_{\lfloor \frac{r}{\pi} \rfloor}$, so in particular we can take $k = \lfloor \frac{r}{\pi} \rfloor$ and the preceding argument holds. On the other hand, $\frac{\alpha-1}{(\alpha-1)^2 + 4\alpha\sin^2(r\cos(\theta))} < \frac{1}{\alpha-1}$, so integrating across this entire region gives us a value of at most $\frac{4}{\sqrt{r}(\alpha-1)^{\frac{3}{4}}}$, while integrating across the rest of the interval $[0, \frac{\pi}{2}]$ gives us a value of at most $\frac{\pi}{2}$, we have shown that the integral across all of the remaining $\mathcal{R}_k$, $k < \lfloor \frac{r}{\pi} \rfloor$, must yield a positive number, and for all other portions of the interval $\frac{\alpha-1}{(\alpha-1)^2 + 4\alpha\sin^2(r\cos(\theta))} < 1$ by design. Also, recall that we established that $r > \frac{\pi}{2}$, so in particular $r > 1$. Thus we have that

$$\int_0^{\frac{\pi}{2}} \frac{\alpha - 1}{(\alpha-1)^2 + 4\alpha\sin^2(r\cos(\theta))} d\theta \leq 4(\alpha-1)^{-\frac{3}{5}} + \frac{\pi}{2} \tag{9}$$

as desired. This establishes that the chromatic number of the odd-distance graph is indeed infinite, as claimed.