TJHSST Senior Research Project
Investigation of Minimal Conjugate Generators for the Symmetric Group and their Cayley Graphs
2007-2008

Jacob Steinhardt

September 7, 2007

Abstract

We investigate subsets of the symmetric group with structure similar to that of a graph. The "trees" of these subsets then lead to minimal highly symmetric generating sets of the symmetric group. We show that there exist generating sets among these with edge-transitive Cayley graphs and investigate them in relation to the Lovasz conjecture.

Keywords: Cayley graph, Lovasz conjecture, Hamiltonian cycle, conjugate generators, symmetric group, quasi-hamiltonicity.

1 Introduction - Purpose and Scope

Note that a graph can be defined as a collection of vertices and edges. Two vertices are adjacent if there exists an edge connecting them, and two vertices v_1 and v_2 are connected if there exists a sequence of adjacent vertices containing v_1 and v_2. On the other hand, consider the following definition: Given a collection of vertices V and a collection of edges E, we can let each element of E act on V as a transposition swapping the two vertices on which E is incident. If we then let multiplication in E extend through the definitions of a group action, E generates a subgroup of the symmetric group acting on V (we denote this subgroup as $< E >$). Then we say that v_1 and v_2 are adjacent if $(v_1\ v_2) \in E$, and that v_1 and v_2 are connected if $(v_1\ v_2) \in < E >$. Additionally, connected components correspond to orbits of V under E. A tree is a minimal generating set of $S_{|V|}$ consisting only of transpositions (thus the fact that trees have $n-1$ vertices corresponds to the fact that it takes $n-1$ transpositions to generate S_n). It is easily verified that these definitions are equivalent.

The algebraic properties of the Cayley graphs arising from this definition are reasonably well-understood. We know that the Lovasz conjecture holds in this case, and also that $Aut(Cay(S_n, T))$ is isomorphic to S_n whenever T represents a tree with trivial automorphism group [2].

The above definition can be generalized. Given a collection of vertices (which, from now on, for convenience, will without loss of generality be N), and a set $T \subset S_n$ in which all elements of T are conjugate (say with conjugacy class $\mathcal{C}$), then we can define elementary notions in a $\mathcal{C}$-graph as follows. $v_1, v_2 \in N$ are adjacent if they have the same orbit under a single element of T. They are *semi-connected* if they have the same orbit under T, and connected if $(v_1\ v_2) \in < T >$ (it is then easy to verify that semi-connectivity and connectivity are equivalence relations). Connected components correspond to subsymmetric groups of $< T >$. A tree is a minimal generating set of S_n with all elements lying in $\mathcal{C}$. It is natural to ask why we add the somewhat artificial-looking stipulation that all elements of T belong to the same conjugacy class. The main reason is that this stipulation is inherent in the construction of a normal graph, where all edges are transpositions. Additionally, without this restriction we get the result that a tree, under our fairly intuitive definition, almost always has 2 edges since $(1\ 2)$ and $(2\ 3\ 4\ \ldots\ n)$ generate S_n.

Although the definitions so far were fairly straightforward, generalizations for other definitions are less obvious. Instead of studying this avenue further, we will characterize $\mathcal{C}$-trees and study some of their properties. However, we will still use the language of graphs for the sake of intuition. We will include approaches to extending the above intuitive generalization to a well-structured system in the section of the paper on open problems.

The goal of the project will be to better understand the algebraic and graph-theoretic properties of these $\mathcal{C}$-trees, especially properties related to the Lovasz conjecture.

2 Background

It is conjectured that every vertex-transitive graph has a Hamiltonian path. A modified version of this conjecture is that every Cayley graph has a Hamiltonian cycle. The original conjecture was initiated by Lovasz, who actually believed it to be false. Laszlo Babai later published a paper attacking the conjecture, giving his own conjecture that there existed Cayley graphs with maximal path length boundable by $c|V|$, for some $c < 1$. The author personally believes the camp of Lovasz and Babai to be the correct one.

However, many weaker versions of the Lovasz conjecture have been shown. For example, with the exception of K_2 and the Peterson graph, which both are known to contain Hamiltonian paths, it has been shown that all graphs Γ for which there exists a transitive subgroup H of $Aut(\Gamma)$ with cyclic commutator of prime-power order contain a Hamiltonian cycle [1]. It has also been shown that there exist small generating sets in every group for which the Lovasz conjecture holds [5], and furthermore that it holds with high probability for generating sets of order $O(\log^5(|G|))$. There are also results for certain classes of generators as well as all abelian groups and graphs with "small" order (e.g. $2p$, $4p$). It is generally agreed that very little progress has been made towards an approach on the general problem.

On the other hand, Gutin and Yeo ([3]) have constructed polynomial time algorithms for checking obstacles to Hamiltonicity. Though these are not very effective on Cayley graphs, we aim to set some groundwork for development of a more effective algorithm specialized to Cayley graphs.

3 Procedures

I need to learn Algebraic Graph Theory, for which I have the book *Algebraic Graph Theory* by Godsil and Royle, as well as read some more papers on partial results and try to find a more effective algorithm than Angluin/Valiant for sparse graphs. I am learning Algebraic Graph Theory independently during 1st period, so this should more or less be taken care of.

I also need to implement the algorithms described below. Once the quasi-hamiltonicity algorithm has been implemented, all computing resources will be helpful as the complexity for k-quasi-hamiltonicity is $O(|E|^{k+1})$ and I would like to check as high as possible for a graph with $|E| = 10^4$.

At this point, significant partial steps to be making are studying the automorphism groups of the given Cayley graphs and their relation to automorphism groups of other graphs related to the generating sets. Also, the characterization of sets with left coset partitions, and an attempt at an analog of the theorem in Godsil and Royle regarding "small" automorphism groups of Cayley graphs generated by an asymmetric tree.

3.1 Testing

I will use random testing on small graphs for all algorithms implemented (this is necessary since any simple validator must be super-polynomial in complexity). Since the implementation itself should not be a substantial part of the project, a detailed timeline is not important. It is, however, important that the programs be flexible, and so care will be taken to allow for such flexibility by allowing separate methods for separable sub-algorithms, such as the DFS and flow-forcing methods in the quasi-hamiltonicity algorithm. This has the advantage of allowing for hopefully easy modification of the algorithms for later specialized purposes.

3.2 Software

Due to requirements of high performance, C++ will be used extensively in the project.

3.3 Algorithms/Programs

The following algorithms will be implemented and used to gather data on relevant Cayley graphs:

1. Angluin and Valiant's algorithm for finding Hamiltonian cycles
2. Gutin and Yeo's algorithm for checking quasi-hamiltonicity
3. An improvement of Gutin and Yeo's algorithm for undirected graphs

I also hope to develop, or start to develop, a more effective version of Gutin and Yeo's algorithm for Cayley graphs.

4 Schedule

During the first quarter, I hope to derive some general properties of the investigated Cayley graphs as well as implement, test, and run Angluin/Valiant and modified Gutin/Yeo.
During the second quarter, I hope to focus more on specific properties of some of the "small" Cayley graphs as well as design specialized Hamiltonicity algorithms for these graphs, the class of graphs studied, or Cayley graphs in general.
During the third quarter, I hope to prove or disprove the Lovasz conjecture for the case of S_{4k-1} generated by two cycles of length $2k$.

5 Expected Results

It is suspected that the Lovasz conjecture is false for S_7 generated by (1234) and (7654). If this is not shown, we at least expect to derive partial results concerning Hamiltonicity of this and similar graphs.

References

[1] Edward Dobson et al. Automorphism groups with cyclic commutator subgroup and hamilton cycles.

[2] Chris Godsil and Gordon Royle. *Algebraic Graph Theory*. Springer, 2004.

[3] Gregory Gutin and Anders Yeo. Quasi-hamiltonicity: a series of necessary conditions for a digraph to be hamiltonian. Technical Report PP-1998-09, 1, 1998.

[4] Roman Nedela and Martin Skoviera. Cayley snarks and almost simple groups. *Combinatorica*, 21:583–590, October, 2001.

[5] Igor Pak and Rados Radoicic. Hamiltonian paths in cayley graphs.

[6] Herbert S. Wilf. *Algorithms and complexity*. Summer, 1994.