Computer Systems Project Proposal Conformal Mapping Using the Schwarz-Christoffel Transform 2007-2008

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Abstract

The Schwarz-Christoffel transform is a conformal mapping from the upper half of the complex plane to a polygonal domain. It allows many physical problems posed on two-dimensional, polygonal regions, such as heat flow, fluid flow, and electrostatics, to be solved numerically. This type of problem cannot generally be solved in closed form; the Schwarz-Christoffel transform provides an exceptionally accurate method of solution. This project will produce a working software unit that efficiently and accurately calculates Schwarz-Christoffel transforms and inverses. The program will incorporate graphical, easy-to-use interfaces and contain resources to aid in solving physical problems. In addition, research into mathematical extensions to the Schwarz-Christoffel transform, such as the inclusion of simple curves, will be conducted.

Keywords: Schwarz-Christoffel transform, conformal mapping, numerical analysis, Laplace's equation, fluid flow, heat flow

1 Scope of Study

The primary and necessary component of this project is the coding of a working and effective (meaning reasonably fast and accurate) program that calculates the Schwarz-Christoffel transform. Although it would not be ideal, this program could constitute the entire project if time constraints become important. Additionally, however, the goal of the project would be to conduct and implement research into extensions of the Schwarz-Christoffel transform, most notably in the inclusion of certain curved domains. Much of this research would be novel, although some preliminary research into the inclusion of circular arcs has been implemented, with moderate success. In addition to the numerical algorithms, a fully functional user interface will be developed to allow users to solve real physical problems quickly and easily.

2 Background and Review of Literature

Basic background information on the Schwarz-Christoffel transform and other conformal mappings, including an outline of the procedure used to generate Schwarz-Christoffel maps, is outlined in the textbook Fundamentals of Complex Analysis by Saff and Snider [2], especially the appendix by Trefethen. In short, the Schwarz-Christoffel map provides a general, analytic map from a simple domain, such as the upper half-plane or the unit disc, to a more complicated polygonal domain, thereby satisfying the Reimann Mapping Theorem for a relatively large class of domains. Its construction involves the evaluation of an integral that can almost never be solved in closed form, requiring various numerical methods, as well as the solution of a moderately large system of nonlinear equations.

Several theses and other papers have been published about the topic of numerical conformal mapping since the 1970s, including experimentation with various target domains and numerical methods. Therefore, the choice of integration routines and equation solvers has been clearly established, although dealing with numerically problematic cases, such as unevenly distributed prevertices and highly irregular polygons, has been dealt with in various ways and no consensus has been reached. Attempted extensions of the Schwarz-Christoffel transform to other smooth boundaries, while mathematically sound, has met with little or no success due to the magnitude of the problem of solving a particular class of integral equations which emerge from the construction, such as Davis in 1978. Particularly specific Schwarz-Christoffel mapping programs have been made by Trefethen in 1979 [3] and Howell in 1990 [1]. The latter developed a very good method of dealing with irregularly shaped polygons. The current state-of-the-art in conformal mapping has been developed under Trefethen and modified several times over the years.

This project would expand upon current research in two ways: firstly, by investigating and implementing curved-boundary domains, and secondly by providing an easy-to-use graphical user interface, which would allow non-specialists to use the Schwarz-Christoffel transform with a minimum of effort.

3 Procedure and Methodology

Various sections of the project will be completed almost independently, including testing procedures, because the project consists of a number of different subproblems that can be attacked individually. The first pieces to be coded are the helper programs, including a class to store and evaluate complex arithmetic and a complex function parser. After this, the program for Gauss-Jacobi quadrature, used to evaluate integrals of the form produced by the Schwarz-Christoffel procedures, will be written, followed by the program for solving a system of nonlinear equations (the so-called "parameter problem"). After these basic tools are completed, projected to be midway through the second quarter, a comprehensive user interface will be developed (to replace the basic structure already in place) and various testing classes will be written to evaluate results. By the end of the second quarter, research into the continuous Schwarz-Christoffel transform should be underway.

In order to solve the Schwarz-Christoffel parameter problem, research will be conducted into various approximate Newton's method schemes. Naive application of Newton's method would be counterproductive for two reasons; first, the parameter problem is constrained by a simple inequality, and must be transferred via a change of variables to a more amenable series of equations, and second, it is unrealistic and in some cases impossible to calculate the Jacobian matrix of the problem. To get around this last problem, various numerical algorithms have been designed to produce successive estimates of the Jacobian matrix; before implementation, one of these methods will be chosen. The continuous-parameter problem, derived by extrapolation to an infinite number of verticies on the target polygon, requires the solution of a certain class of integral equations. Here methods are more speculative; however, similar equations have been solved for different problems in the past with reasonable time and error.

The language in which the user interfaces and driver programs will be written is Java. For certain numerical methods, where speed is a large issue and bottlenecks may arise, C may be used; however, preliminary results suggest that this will not be necessary, and the entire project may well be written in Java. First testing will all be completed in Java, including speed analysis, and will then be translated if necessary.

The majority of testing of the program will be specific to a single numerical routine. To test each routine, problems will be chosen that have closed-form, calculable solutions, and the answers will be worked out ahead of time on paper or on a program such as MATLAB, using their quadrature or equation-solving routines. After accuracy, runtime will be the next most important issue, and a significant portion of testing will be timed for a given error tolerance in different scenarios.

Once the entire basic routine is complete, a program specifically designed to track approximate error propagation and runtimes of each of the components will be developed. For the majority of the routines used in the program, strict error bounds can be calculated, and for the remaining algorithms error can be accurately estimated. For all subproblems, as well as the entire routine, plots of runtime versus precision will be generated to examine the efficacy of each routine. For the final program, random polygonal generation will be used to dynamically test the program for a range of inputs.

4 Expected Results

Expected results from this project include the production of visual numerical Schwarz-Christoffel conformal maps in a reasonable time period, defined as on the order of several seconds. The project should also include easy-to-use interfaces and significant aids to solving related problems in physics. After the basic problem has been implemented, extensions to the ordinary Schwarz-Christoffel program will be designed, such as the development of a Schwarz-Christoffel based procedure for extension into arbitrary (nonpolygonal) domains. The last feature would be almost entirely novel, and could require a significant amount of both processing time and mathematical sophistication above and beyond the basic problem.

Error bounds and runtime data on all parts of the program, including the entire program as a whole, will be collected and displayed, with the goal of pointing towards realistic and efficient tolerance levels for each subproblem. Presentation of the final project will be done mostly via the program itself; that is, an integral part of the project is the effective display of the information contained in the problem. The completed program will be useful on several levels: as a teaching aid, and as a tool for researchers solving certain equations on polygonal regions. Once the basic Schwarz-Christoffel problem is numerically solved, the program can form an easy basis for testing research in numerical analysis and mathematics that deals with improving or expanding the Schwarz-Christoffel transform.

References

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