Efficient Computation of Homology Groups of Simplicial Complexes Embedded in Euclidean Space

Brian Hamrick

Thomas Jefferson High School for Science and Technology Computer Systems Lab

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Introduction

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- Simplices

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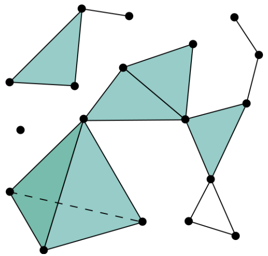

http://commons.wikimedia.org/wiki/File:Simplicial_complex_example.svg

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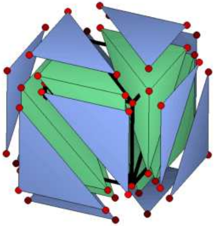

http://www.math.tu-berlin.de/polymake/apps/topaz/images/pile_with_boundary.gif

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