On the Incremental Computation of Simplicial Homology of Triangulated Surfaces

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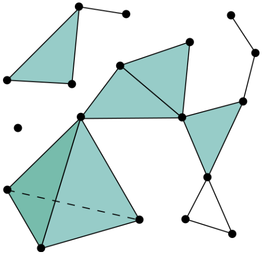
Introduction

- Simplices
- Simplicial Complexes
- Simplicial Homology



http://commons.wikimedia.org/wiki/File:Simplicial_complex_example.svg

Methods

- Matrix reduction to Smith normal form
- Incremental methods
- Geometric methods

Our Method

- Incremental
- Based on the Mayer-Vietoris Sequence

$$\cdots \to H_{n+1}(X) \xrightarrow{\partial_*} H_n(A \cap B) \xrightarrow{(i_*, j_*)} H_n(A) \oplus H_n(B) \xrightarrow{k_* - l_*} H_n(X) \xrightarrow{\partial_*} H_{n-1}(A \cap B) \to \cdots \to H_0(A) \oplus H_0(B) \xrightarrow{k_* - l_*} H_0(X) \to 0$$

Incremental Method

- Idea: Add one simplex at a time, updating homology groups as you go.
- A is our previous simplicial complex K_{i-1}, B is the new simplex σ_i, X is the new simplicial complex K_i.

$$\cdots \to H_{n+1}(K_i) \to H_n(\partial\sigma_i) \to H_n(K_{i-1}) \to H_n(K_i) \to \cdots$$

Two Dimensional Simplices

Three operations:

- Add a vertex
- Add an edge
- Add a triangle

Setup for the Algorithm

Variables to keep track of:

- The fundamental cycles S_i corresponding to σ_i.
- The generators G_j for H_k and their torsion coefficients T_j.
- The representation for each S_i in terms of the generators G_j.

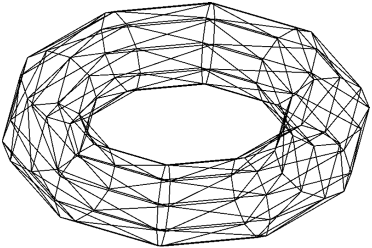
Computing Quotient of Homology Group

- Adding a single relation: $\partial\sigma_i = Z = \sum e_i G_i = 0$.
- Smith Normal Form of $\begin{pmatrix} T_1 & 0 & 0 & \cdots & 0 \\ 0 & T_2 & 0 & \cdots & 0 \\ 0 & 0 & T_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & T_n \\ e_1 & e_2 & e_3 & \cdots & e_n \end{pmatrix}$
- Column operations change generators and thus representations of the fundamental cycles
- Resulting diagonal entries are new torsion coefficients.

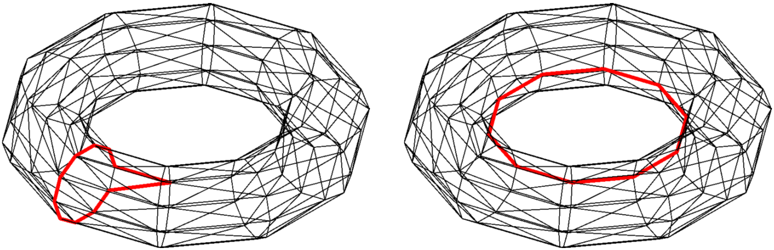
Performance Analysis

- Approximately one Smith Normal Form computation for each simplex.
- Sparse matrix
- Computation at each step is at most $O(\sum T_i)$ row and column operations
- $O(N)$ row and column operations per update for orientable surfaces

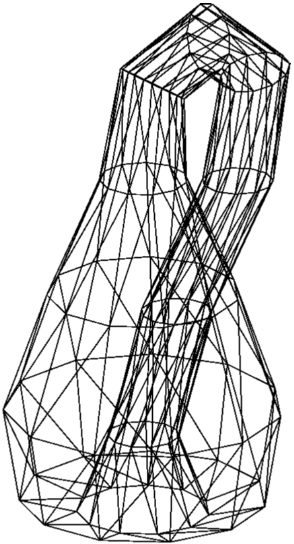
Torus



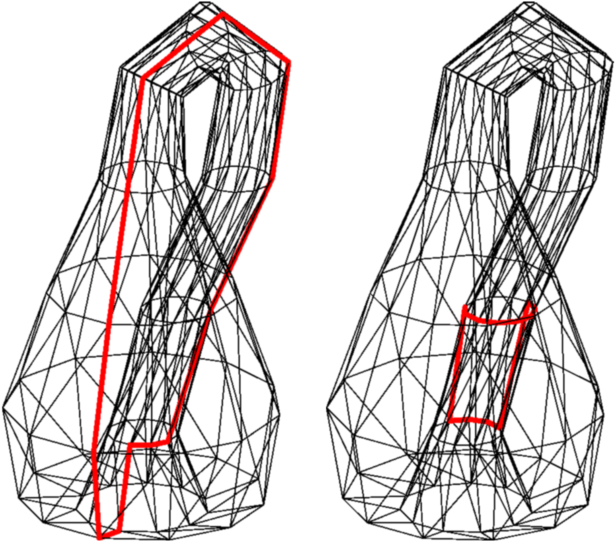
Torus - Results



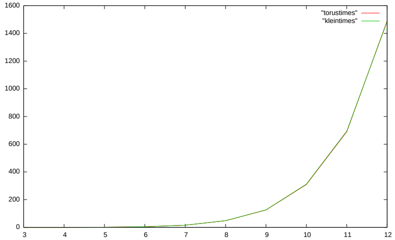
Klein Bottle



Klein Bottle - Results



Total Times



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"kleintimes"
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6
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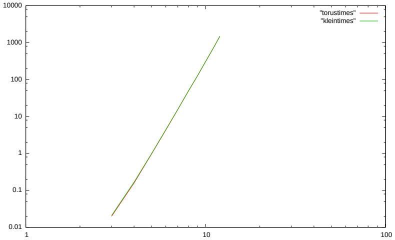
11
12



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"kleintimes"
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100
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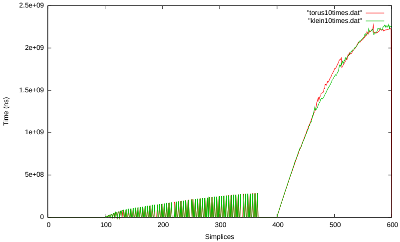
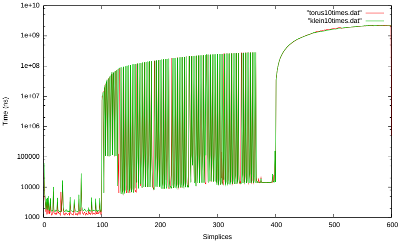
0.01
1
10
100

Update Times





Future Work

- Application to image processing
- Optimization of implementation
- Removal of constraints