On the Incremental Computation of Simplicial Homology

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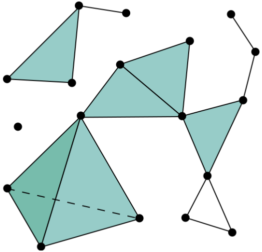
Introduction

- Simplices
- Simplicial Complexes
- Singular Homology
- Simplicial Homology



http://commons.wikimedia.org/wiki/File:Simplicial_complex_example.svg

Homology

Homology

Computed from a chain complex of abelian groups

$$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_4} C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

- $\partial_n \circ \partial_{n+1} = 0$
- $\mathrm{Im}(\partial_{n+1}) \subseteq \mathrm{Ker}(\partial_n)$
- $\mathrm{Im}(\partial_{n+1}) \trianglelefteq \mathrm{Ker}(\partial_n)$
- The homology groups are defined as $H_n = \mathrm{Ker}(\partial_n)/\mathrm{Im}(\partial_{n+1})$
- Singular Homology
- Simplicial Homology

Methods

- Matrix reduction to Smith normal form
- Incremental methods
- Geometric methods

Our Method

- Incremental

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- Incremental
- Based on the Mayer-Vietoris Sequence

The Mayer-Vietoris Sequence

An exact sequence relating homology groups. If A and B are two spaces whose interiors cover X, then

$$\cdots \to H_{n+1}(X) \stackrel{\partial_*}{\to} H_n(A \cap B) \stackrel{(i_*, j_*)}{\longrightarrow} H_n(A) \oplus H_n(B) \stackrel{k_* - l_*}{\longrightarrow} H_n(X) \stackrel{\partial_*}{\to}$$
$$\stackrel{\partial_*}{\to} H_{n-1}(A \cap B) \to \cdots \to H_0(A) \oplus H_0(B) \stackrel{k_* - l_*}{H}{}_0(X) \to 0$$

is an exact sequence.

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$$\cdots \to H_{n+1}(K_i) \to H_n(\partial\sigma_i) \to H_n(K_{i-1}) \to H_n(K_i) \to \cdots$$

One Dimensional Complexes

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- Add a vertex
- Add an edge
  - Join two components (reduce H_0)
  - Create a cycle (expand H_1)

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  - For two dimensional complexes where each one-simplex has at most two adjacent two-simplices, can use a floodfill.

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- How to find cycles?
    - For two dimensional complexes where each one-simplex has at most two adjacent two-simplices, can use a floodfill.
    - Not sure for higher dimensions.