TJ USAMO Practice 11 - Number Theory II

Varsity Math Team

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1. The Cauchy-Davenport Lemma states that for any prime p, two sets A and B of integers, and the set $A + B := \{a + b \,|\, a \in A, b \in B\}$, we have either $\text{size}(A + B) \geq \text{size}(A) + \text{size}(B) - 1$ or $\text{size}(A + B) = p$, where $\text{size}(X)$ denotes the number of distinct residues obtained when each element of X is taken modulo p. Prove the Cauchy-Davenport Lemma for the special case $\text{size}(B) = 2$.

2. Show that there are no integers $n_1, n_2, \ldots, n_{31}$ such that
$$n_1{}^8 + n_2{}^8 + \cdots + n_{31}^8 = 2005 \cdot 10^5$$

3. Let p be an odd prime, and let a and b be integers relatively prime to p. Prove that there exist integers x and y such that $1 \leq x, |y| \leq \lfloor \sqrt{p} \rfloor$ and $ax + by \equiv 0 \pmod{p}$.

4. S is a set of $n - 1$ integers. Given that some two elements of S are distinct when taken modulo n, show that there exists a non-empty subset S' such that the sum of the elements of S' is divisible by n.

5. (Erdös-Ginsburg-Ziv) Show that among any $2n - 1$ integers, we can find n integers $a_1, \ldots, a_n$ such that the sum of the a_i is divisible by n.