TJ USAMO Practice 12 - Contest 2 Solutions

Varsity Math Team

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1. Show that on any tetrahedron we can find a vertex V such that the lengths of the 3 edges with V as an endpoint can be the sides of a triangle.

   Solution.

   Label the vertices of a tetrahedron $ABCD$ such that $\overline{AB}$ is at least as long as any other edge. Suppose that neither the edges with common endpoint at A nor the edges with common endpoint at B can be arranged to make a triangle. As $AB > \max(AC, AD, BC, BD)$, it must be that $AB > AC + AD$ and $AB > BC + BD$, hence $2AB > AC + AD + BC + CD$. But, by the triangle inequality, $AB < AC + CB$ and $AB < AD + DB$, which, when added, imply $2AB < AC + AD + BC + BD$. Contradiction.

2. Let a, b, and c be non-zero integers such that both

   $$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \quad \text{and} \quad \frac{a}{c} + \frac{c}{b} + \frac{b}{a}$$

   are integers. Prove that $|a| = |b| = |c|$.

   Solution.

   Let $d = \gcd(a, b, c)$, and define $a' = \frac{a}{d}, b' = \frac{b}{d}$, and $c' = \frac{c}{d}$. Clearly, the result holds for a, b, and c if and only if it holds for a', b', and c'. Suppose that p is the least prime that divides $a'b'c'$. Let d_A denote the number of times that p divides a', and define d_B and d_C analogously. WLOG, $d_A \geq d_B \geq d_C$. Since no prime divides a', b', and c', it must be that $d_C = 0$ and $d_A \geq 1$. Now, if $\frac{a'}{b'} + \frac{b'}{c'} + \frac{c'}{a'} = \frac{a'^2c'+b'^2a'+c'^2b'}{a'b'c'}$ is an integer, it must be that $a'b'c'$ divides $a'^2c' + b'^2a' + c'^2b'$. This can happen only if $p^{d_A+d_B}$ divides $a'^2c' + b'^2a' + c'^2b' \iff p^{d_A+d_B}$ divides c'^2b'. But because p divides c'^2b' exactly d_B times, it must be that $d_A = 0$, contradiction. Therefore, there is no prime that divides $a'b'c'$, which implies that $|a| = |b| = |c|$.

3. Let ABC be a scalene triangle, and P a point on the angle bisector of $\angle BAC$. Let the extensions of $\overline{PB}$ and $\overline{PC}$ intersect with the extensions of $\overline{AC}$ and $\overline{AB}$ at D and E respectively. Show that P lies on the circumcircle of ABC if and only if $EP \cdot DC = DP \cdot EB$.

Solution.

Menelaus applied to triangle ACE and line DB yields $\frac{AD}{DC}\frac{CP}{PE}\frac{EB}{BA} = -1$, and the angle bisector theorem yields $\frac{AD}{BA} = \frac{DP}{PB}$. Substituting and clearing the denominator yields the identity

$$CP \cdot EB \cdot DP = DC \cdot PE \cdot PB$$

It is clear that $EP \cdot DC = DP \cdot EB$ if and only if $CP = PB$. Let Q denote the intersection (other than A) of the bisector of angle A and the circumcircle of ABC. It is clear by the Law of Sines that $CQ = QB$, but as $CA \neq AB$, it must be that $\overline{AP}$ is not parallel to the perpendicular bisector of $\overline{BC}$. Thus, Q is the unique point on the bisector of $\angle A$ with $CQ = QB$. The desired result follows.

Alternate Solution.

By the Law of Sines, $\frac{DP}{DC} = \frac{\sin\angle DCP}{\sin\angle CPD}$ and $\frac{BE}{EP} = \frac{\sin\angle EPB}{\sin\angle PBE}$. Since $\angle CPD \cong \angle EPB$, it follows that $EP \cdot DC = DP \cdot EB$ iff $\sin\angle DCP = \sin\angle PBE$ iff $\sin\angle PCA = \sin\angle ABP$. There are two cases in which $\sin\angle PCA = \sin\angle ABP$. If $\angle PCA \cong \angle ABP$, then by angle-angle-side congruence, $\triangle CAP \cong \triangle BAP$, which implies that $AB = AC$, contradiction. So we must have the second case, that $\angle PCA$ and $\angle ABP$ are supplementary. This happens if and only if $CABP$ is a cyclic quadrilateral, Q.E.D.