TJ USAMO Practice 3 - Inequalities I

Varsity Math Team

October 14, 2004

Which theorem is Botchy's theorem?

1 The Tools

These are some of the inequalities that you are allowed to assume while proving contest inequalities:

- **The Trivial Inequality** - The inequality $r^2 \geq 0$ for any real number r. Equality holds if and only if $r = 0$.

- **Arithmetic Mean - Geometric Mean - Harmonic Mean** - Let n be any positive integer, and let $a_1, a_2, \ldots, a_n$ be any non-negative real numbers. Then

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

  That is, AM $\geq$ GM. Also, if $a_1, a_2, \ldots, a_n$ are positive,

$$\sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

  That is, GM $\geq$ HM. Equality in each inequality holds if and only if $a_1 = a_2 = \cdots = a_n$.

- **Cauchy's Inequality** - Let n by any positive integer. Then, for any sequences $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ of real numbers, we have

$$\left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + b_n^2\right) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2$$

  Equality holds here if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$ (where we ignore $\frac{0}{0}$'s.)

- **Schur's Inequality** - Let r be a positive real number. For all non-negative reals a, b, and c, we have

$$a^r(a-b)(a-c) + b^r(b-c)(b-a) + c^r(c-a)(c-b) \geq 0$$

  With equality if and only if $a = b = c$ or $a = 0$ and $b = c$.

2 The Problems

Using the previously mentioned inequalities, prove the following:

0. For any real x

$$x^4 - x^2 - 2x + 2 \geq 0$$

1. For any positive integer n, and positive reals $a_1, a_2, \ldots, a_n$ with $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = n$,

$$\frac{1}{1+a_1} + \frac{1}{1+a_2} + \cdots + \frac{1}{1+a_n} \leq \frac{n}{2}$$

2. (USAMO 1997) Show that for all positive reals a, b, and c,

$$\frac{1}{a^3+b^3+abc} + \frac{1}{b^3+c^3+abc} + \frac{1}{c^3+a^3+abc} \leq \frac{1}{abc}$$

3. (MOP 2002) a, b, and c are the sides of an acute triangle. Show that

$$(a+b+c)\left(a^2+b^2+c^2\right)\left(a^3+b^3+c^3\right) > 4a^6 + 4b^6 + 4c^6$$

4. (Inspired by APMO 2004 #5) For all positive reals a, b, and c,

$$a^2b^2c^2 + a^2 + b^2 + c^2 + 2 \geq 2ab + 2bc + 2ca$$

5. (MOP 2002) Show that for any positive reals a, b, c, and r such that $r \geq \frac{2}{3}$, we have

$$\left(\frac{a}{b+c}\right)^r + \left(\frac{b}{c+a}\right)^r + \left(\frac{c}{a+b}\right)^r \geq \frac{3}{2^r}$$