

# TJ USAMO Practice 5 - Induction

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Here are some powerful methods to prove a proposition  $P(n)$  for all positive integers  $n$ .

- **Induction:** Prove  $P(1)$  and that if  $P(n)$  is true, then  $P(n+1)$  is also true.
- **Strong Induction:** Prove  $P(1)$  and that if  $P(1), P(2), \dots, P(n)$  are all true, then  $P(n+1)$  is also true.
- **Well Ordering Principle:** Take the smallest  $n$  such that  $P(n)$  is false. Then show that, for some  $k < n$ ,  $P(k)$  is false (which could require proving a base case). This is a contradiction, so  $P(n)$  is true for all  $n$ .

1. Prove

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

for all positive integers  $n$ .

2. Let  $a_n$  denote the number of non-empty sets  $S \subseteq \{1, 2, 3, \dots, n\}$  such that all elements of  $S$  have the same parity and  $k \geq 2|S|$  for all  $k \in S$ , where  $|S|$  is the number of elements in  $S$ . Prove that

$$a_{2m-1} = 2(F_{m+1} - 1) \text{ and } a_{2m} = F_{m+3} - 2,$$

for all  $m \geq 1$ , where  $F_k$  is the  $k$ th Fibonacci number.

3. Prove that there exist positive integers  $a_1, a_2, \dots, a_n$  such that  $|a_i - a_j|$  divides  $a_i + a_j$  for all  $i \neq j$ .
4. A country contains several cities, some pairs of which are connected by flights from one of  $k$  airlines. For each airline, every two flights offered contain a common endpoint. Show that it is possible to partition the cities into  $k+2$  groups such that no two cities in the same group are connected by a flight.