

TJ USAMO Practice 7

Varsity Math Team

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1. Let \mathbb{N} denote the set of all positive integers, and let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function such that $f(m) > f(n)$ for all $m > n$. For each positive integer m , let $g(m)$ denote the number of positive integers n such that $f(n) < m$. Express

$$\sum_{i=1}^{f(n)} g(i) + \sum_{k=1}^n f(k)$$

in closed form.

2. Form a 2003×2005 screen by unit screens. There are more than 2004×2002 unit screens which are *on*. In any 2×2 screen, as soon as there are 3 unit screens which are *off*, the 4th screen will be turned off automatically. Prove that the whole screen will never be totally off.
3. Let D_1, D_2, \dots, D_n be closed discs in the plane. Suppose that every point in the plane is covered by at most 2003 discs. Prove that there exists a disc which intersects at most 14020 other discs. (A closed disc is a region bounded by a circle, taken jointly with this circle.)