TJ USAMO Practice 9 - TJMO Contest 1

Varsity Math Team

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1. Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of positive integers such that

$$x_1 = 1 \quad (1)$$
$$x_i < x_{i+1} \quad \text{for all positive integers } i. \quad (2)$$
$$x_{n+1} \leq 2n \quad \text{for each } n \geq 1. \quad (3)$$

   Show that for any integer k, there exist positive integers i and j such that $k = x_i - x_j$.

2. ABC is a triangle and M is the midpoint of $\overline{BC}$. D is a point on $\overline{BC}$. Let O_1 and O_2 denote the circumcenters of triangles ADB and ACD respectively. Let X denote the intersection of $\overline{O_1O_2}$ and the perpendicular bisector of $\overline{AM}$. Show that X is the midpoint of $\overline{O_1O_2}$.

3. Show that, for all positive reals a, b, and c such that $a+b \geq c$; $b+c \geq a$; and $c+a \geq b$, we have

$$2a^2(b+c) + 2b^2(c+a) + 2c^2(a+b) \geq a^3 + b^3 + c^3 + 9abc$$

   When does equality hold?