Solutions to TJMO Contest 1

Varsity Math Team

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1. Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of positive integers such that

$$\begin{aligned} x_1 &= 1 && (1) \\ x_i &< x_{i+1} \quad \text{for all positive integers } i. && (2) \\ x_{n+1} &\leq 2n \quad \text{for each } n \geq 1. && (3) \end{aligned}$$

Show that for any integer k, there exist positive integers i and j such that $k = x_i - x_j$.

Solution.

The case $k = 0$ is evident by $i = j \Longrightarrow x_i = x_j \iff x_i - x_j = 0$. Now, for any $k > 0$ if there exist i and j such that $x_i - x_j = k$, then for $k' = -k < 0$ we have satisfactory i' and j' given by $i' = j$ and $j' = i$. Now we need only prove the claim for $k \geq 1$.

The given constaints imply that $1 = x_1 < x_2 < x_3 < \cdots < x_{k+1} <= 2k$. Partition the set $S = \{1, 2, 3, \ldots, 2k\}$ into the subsets $T_m = \{m, m + k\}$ for $m = 1, 2, \ldots, k$. Now, $x_1, x_2, \ldots, x_{k+1}$ are $k + 1$ distinct values chosen from S, but by the pigeon hole principle, there exist distinct integers $i > j$ such that $x_i = max(T_m)$ and $x_j = min(T_m)$ for some $1 \leq m \leq k$. Our claim is accordingly established by noting that the difference $max(T_m) - min(T_m) = k$ for each m.

2. ABC is a triangle and M is the midpoint of $\overline{BC}$. D is a point on $\overline{BC}$. Let O_1 and O_2 denote the circumcenters of triangles ADB and ACD respectively. Let X denote the intersection of $\overline{O_1O_2}$ and the perpendicular bisector of $\overline{AM}$. Show that X is the midpoint of $\overline{O_1O_2}$.

Solution.

WLOG, we assume that $D \in \overline{BM}$. Let P, Q, and X' denote the feet of the perpendiculars from O_1, O_2, and X to $\overline{BC}$, respectively. Because $\overline{O_1P} \parallel \overline{XX'} \parallel \overline{O_2Q}$, X is the midpoint of $\overline{O_1O_2}$ if and only if X' is the midpoint of $\overline{PQ}$. $\overline{O_1O_2}$ is the perpendicular bisector of $\overline{AD}$, hence, X is the circumcenter of triangle AMD, which implies that $DX' = X'M$. Accordingly, $PX' = X'Q \iff PD = MQ$.

Because dilation of scale factor two about C maps Q and M to D and B respectively, it must be that $BD = 2QM$. But $BD = 2PD$, hence $PD = MQ$, establishing our claim.

3. Show that, for all positive reals a, b, and c such that $a+b \geq c$; $b+c \geq a$; and $c+a \geq b$, we have
$$2a^2(b+c) + 2b^2(c+a) + 2c^2(a+b) \geq a^3 + b^3 + c^3 + 9abc$$
When does equality hold?

Solution.

Definitions - We write $\sum_{cyc} f(x,y,z)$ for $f(x,y,z)+f(y,z,x)+f(z,x,y)$ and $\sum_{sym} f(x,y,z)$ for $f(x,y,z) + f(x,z,y) + f(y,x,z) + f(y,z,x) + f(z,x,y) + f(z,y,x)$.

Consider the variables $x = \frac{b+c-a}{2}$, $y = \frac{c+a-b}{2}$, and $z = \frac{a+b-c}{2}$. It is clear from the givens that $x, y, z \geq 0$, but we may write $a = y+z, b = z+x, c = x+y$. Substituting, we write the equivalent

$$\begin{aligned} 2(y+z)^2(2x+y+z) + 2(z+x)^2(2y+z+x) &+ 2(x+y)^2(2z+x+y) \geq \\ (y+z)^3 &+ (z+x)^2 + (x+y)^3 + 9(y+z)(z+x)(x+y) \\ \iff \sum_{sym}(2x^3 + 10x^2y + 4xyz) &\geq \sum_{sym}\left((x^3+3x^2y) + 9(x^2y + \frac{xyz}{3})\right) \\ \iff \sum_{sym} x^3 + xyz &\geq \sum_{sym} 2x^2y \\ \iff x^3 + y^3 + z^3 + 3xyz &\geq x^2(y+z) + y^2(z+x) + z^2(x+y) \end{aligned}$$

This is true because it is a special case of Schur's inequality. We have equality iff $x = y = z$; $x = y, z = 0$; $y = z, x = 0$; or $z = x, y = 0$. Translated back into a, b, and c, these equality cases are $a = b = c$; $a = b, c = a+b$; $b = c, a = b+c$; and $c = a, b = c+a$.