Exponential Functions and Logarithms

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1 Exponential Functions

The basic exponential function is $f(x) = Ca^x$. We generally assume $C, a > 0$. We have *exponential growth* and *decay* where $a > 1$ and $0 < a < 1$ respectively. $a = 1$ defines the constant $f(x) = C$.

Exponential functions are evaluated in the same way as any other function, parameters are substituted as provided and we simply our result. Graphing works in the same manner as previous f. That is, transformations follow the same rules. Replacing x with kx results in a squish of factor k along the x-dimension. Similarly, multiplying f by c results in a vertical stretch of c. Adding a constant j results in a vertical shift of j, where a negative vertical shift is a downward shift. Finally, replacing x with $x+i$ causes a *leftward* shift of i units.

It is also of significance that we have $a^{bx} = (a^b)^x$ and $a^{x+c} = a^x * a^c = Ca^x$. The significance of these two facts allows us to map general exponential functions onto each other. That is, if $f(x) = C_1a^x$ and $g(x) = C_2b^x$, then there is a way to obtain g as a function of f.

When solving for x in exponential functions, we typically resort to guess and check, as in the case $2^x = 512$. We may try $x = 6$ and get 64, then $x = 8$ and get 256, then try $x = 9$ and obtain our answer. This method fails for most random cases. What x solves $3^x = 8$? Plugging in small integers shows that $1 < x < 2$, but guessing decimals becomes slow and useless. In such a case, we must fall back on *logarithms*.

2 Logarithms

NOTE: From here on out, all rules assume $a > 0, a \neq 1$ and $x > 0$.
$log_a x$ is read “Log base a of x”. We define $log_a x = y$ (logarithmic form) such that $a^y = x$ (exponential form). So, in our problem $3^x = 8$, we have $x = log_3 8$. From the definition of

$log_a x$, we obtain several important rules:

$i)\ log_a a = 1$
$ii)\ log_a 1 = 0$
$iii)\ log_a a^x = x = a^{log_a x}$

From the nature of exponents, we obtain several other less specific indentities:

Product Property: $log_a bc = log_a b + log_a c$
Quotient Property: $log_a \frac{b}{c} = log_a b - log_a c$
Power Property: $log_a b^c = c\ log_a b$

By falling back on the definition, we see that raising exponential form to a power k yields $log_a b = log_{(a^k)}(b^k)$, and by using a reversal of the power property, $log_{(a^k)} b = \frac{log_a b}{k}$. This identity allows us to remove exponents from the base of a logarithm, and by choosing $k = log_a c$, we obtain the change of base identity, $log_a b = \frac{log_c b}{log_c a}$. With this identity, we get a product of logs identity: $log_a b * log_b c = log_a c$.

There is one last identity that gives us a degree of freedom when dealing with logarithms in exponents. After establishing that $log_{(a^k)}(b^k) = log_a b$, we use this to derive this identity:

$$a^{log_b c} = a^{log_{(b^{log_b a})}(c^{log_b a})} = a^{log_{(a)}(c^{log_b a})} = c^{log_b a}$$

Armed with these properties, we can solve a number of exponential and logarithmic problems. Here are some examples:

1. Expand as far as possible: $log_3 2x^2 = log_3 2 + log_3 x^2 = log_3 2 + 2\ log_3 x$
2. Expand as far as possible: $log_7 \frac{2x^3y^2}{14} = log_7 \frac{x^3y^2}{7} = log_7 x^3y^2 - log_7 7 = 3\ log_7 x + 2\ log_7 y - 1$
3. Condense as much as possible: $log_4 5 + 2\ log_4 x + 4\ log_4 y = log_4 5 + log_4 x^2 + log_4 y^4 = log_4 5x^2y^4$
4. Simplify: $2(log_7 18 - log_7 3) - \frac{1}{2} log_{49} 64 = 2\ log_7 6 - log_{49} 8 = log_7 36 - log_{49} 8 = log_7 6 - \frac{1}{2} log_7 8 = log_7 \frac{6}{\sqrt{8}} = log_7 \frac{3\sqrt{2}}{2}$
5. Evaluate: $log_4 3 * log_9 7 * log_7 2 = \frac{1}{2} log_2 3 * \frac{1}{2} log_3 7 * log_7 2 = \frac{1}{4} log_2 7 * log_7 2 = \frac{1}{4}$