Mock AIME

Thomas Mildorf

3:00 - 6:00 PM EST, July 31, 2004

1. Let S denote the sum of all of the three digit positive integers with three distinct digits. Compute the remainder when S is divided by 1000.

2. If $x^2 + y^2 - 30x - 40y + 24^2 = 0$, then the largest possible value of $\frac{y}{x}$ can be written as $\frac{m}{n}$, where m and n are relatively prime, positive integers. Determine $m + n$.

3. A, B, C, D, and E are collinear in that order such that $AB = BC = 1$, $CD = 2$, and $DE = 9$. If P can be any point in space, what is the minimum possible value of $AP^2 + BP^2 + CP^2 + DP^2 + EP^2$?

4. When $1 + 7 + 7^2 + \cdots + 7^{2004}$ is divided by 1000, a remainder of N is obtained. Determine the value of N.

5. Let a and b be the two real values of x for which
$$\sqrt[3]{x} + \sqrt[3]{20 - x} = 2$$
The smaller of the two values can be expressed as $p - \sqrt{q}$, where p and q are integers. Compute $p + q$.

6. A paperboy delivers newpapers to 10 houses along Main Street. Wishing to save effort, he doesn't always deliver to every house, but to avoid being fired he never misses three consecutive houses. Compute the number of ways the paperboy could deliver papers in this manner.

7. Let N denote the number of permutations of the 15-character string AAAABBBBBCCCCCC such that

None of the first four letters is an A. (1)

None of the next five letters is a B. (2)

None of the last six letters is a C. (3)

Find the remainder when N is divided by 1000.

8. $ABCD$, a rectangle with $AB = 12$ and $BC = 16$, is the base of pyramid $\mathcal{P}$, which has a height of 24. A plane parallel to $ABCD$ is passed through $\mathcal{P}$, dividing $\mathcal{P}$ into a frustum $\mathcal{F}$ and a smaller pyramid $\mathcal{P}'$. Let X denote the center of the circumsphere of $\mathcal{F}$, and let T denote the apex of $\mathcal{P}$. If the volume of $\mathcal{P}$ is eight times that of $\mathcal{P}'$, then the value of XT can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute the value of $m + n$.

9. p, q, and r are three non-zero integers such that $p + q + r = 26$ and

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{360}{pqr} = 1$$

Compute pqr.

10. $ABCDEFG$ is a regular heptagon inscribed in a unit circle centered at O. l is the line tangent to the circumcircle of $ABCDEFG$ at A, and P is a point on l such that $\triangle AOP$ is isosceles. Let p denote value of $AP \cdot BP \cdot CP \cdot DP \cdot EP \cdot FP \cdot GP$. Determine the value of p^2.

11. Let S denote the value of the sum

$$\sum_{n=0}^{668} (-1)^n \binom{2004}{3n}$$

Determine the remainder obtained when S is divided by 1000.

12. $ABCD$ is a rectangular sheet of paper. E and F are points on $\overline{AB}$ and $\overline{CD}$ respectively such that $BE < CF$. If $BCFE$ is folded over $\overline{EF}$, C maps to point C' on $\overline{AD}$ and B maps to B' such that $\angle AB'C' \cong \angle B'EA$. If $AB' = 5$ and $BE = 23$, then the area of $ABCD$ can be expressed as $a + b\sqrt{c}$ square units, where a, b, and c are integers and c is not divisible by the square of any prime. Compute $a + b + c$.

13. A sequence $\{R_n\}_{n \geq 0}$ obeys the recurrence $7R_n = 64 - 2R_{n-1} + 9R_{n-2}$ for any integers $n \geq 2$. Additionally, $R_0 = 10$ and $R_1 = -2$. Let

$$S = \sum_{i=0}^{\infty} \frac{R_i}{2^i}$$

S can be expressed as $\frac{m}{n}$ for two relatively prime positive integers m and n. Determine the value of $m + n$.

14. Wally's Key Company makes and sells two types of keys. Mr. Porter buys a total of 12 keys from Wally's. Determine the number of possible arrangements of Mr. Porter's 12 new keys on his keychain (Where rotations are considered the same and any two keys of the same type are identical.)

15. Triangle ABC has an inradius of 5 and a circumradius of 16. If $2\cos B = \cos A + \cos C$, then the area of triangle ABC can be expressed as $\frac{a\sqrt{b}}{c}$, where a, b, and c are positive integers such that a and c are relatively prime and b is not divisible by the square of any prime. Compute $a + b + c$.