Mock AIME 2

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1. Compute the largest integer k such that 2004^k divides $2004!$.

2. x is a real number with the property that $x + \frac{1}{x} = 3$. Let $S_m = x^m + \frac{1}{x^m}$. Determine the value of S_7.

3. In a box, there are 4 green balls, 4 blue balls, 2 red balls, a brown ball, a white ball, and a black ball. These balls are randomly drawn out of the box one at a time (without replacement) until two of the same color have been removed. This process requires that at most 7 balls be removed. The probability that 7 balls are drawn can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

4. Let $S := \{5^k | k \in \mathbb{Z}, 0 \leq k \leq 2004\}$. Given that $5^{2004} = 5443 \cdots 0625$ has 1401 digits, how many elements of S begin with the digit 1?

5. Let S be the set of integers $n > 1$ for which $\frac{1}{n} = 0.d_1d_2d_3d_4\ldots$, an infinite decimal that has the property that $d_i = d_{i+12}$ for all positive integers i. Given that 9901 is prime, how many positive integers are in S? (The d_i are digits.)

6. ABC is a scalene triangle. Points D, E, and F are selected on sides BC, CA, and AB respectively. The cevians AD, BE, and CF concur at point P. If $[AFP] = 126, [FBP] = 63$, and $[CEP] = 24$, determine the area of triangle ABC.

7. Anders, Po-Ru, Reid, and Aaron are playing Bridge. After one hand, they notice that all of the cards of two suits are split between Reid and Po-Ru's hands. Let N denote the number of ways 13 cards can be dealt to each player such that this is the case. Determine the remainder obtained when N is divided by 1000. (Bridge is a card game played with the standard 52-card deck.)

8. Determine the remainder obtained when the expression

$$2004^{2003^{2002^{2001}}}$$

is divided by 1000.

9. Let

$$\left(1 + x^3\right)\left(1 + 2x^{3^2}\right)\cdots\left(1 + kx^{3^k}\right)\cdots\left(1 + 1997x^{3^{1997}}\right) = 1 + a_1x^{k_1} + a_2x^{k_2} + \cdots + a_mx^{k_m}$$

where $a_i \neq 0$ and $k_1 < k_2 < \cdots < k_m$. Determine the remainder obtained when a_{1997} is divided by 1000.

10. $ABCDE$ is a cyclic pentagon with $BC = CD = DE$. The diagonals AC and BE intersect at M. N is the foot of the altitude from M to AB. We have $MA = 25, MD = 113$, and $MN = 15$. The area of triangle ABE can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Determine the remainder obtained when $m + n$ is divided by 1000.

11. α, β, and γ are the roots of $x(x - 200)(4x + 1) = 1$. Let

$$\omega = \tan^{-1}(\alpha) + \tan^{-1}(\beta) + \tan^{-1}(\gamma)$$

The value of $\tan(\omega)$ can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers. Determine the value of $m + n$.

12. $ABCD$ is a cyclic quadrilateral with $AB = 8, BC = 4, CD = 1$, and $DA = 7$. Let O and P denote the circumcenter and intersection of AC and BD respectively. The value of OP^2 can be expressed as $\frac{m}{n}$, where m and n are relatively prime, positive integers. Determine the remainder obtained when $m + n$ is divided by 1000.

13. $P(x)$ is the polynomial of minimal degree that satisfies

$$P(k) = \frac{1}{k(k+1)}$$

for $k = 1, 2, 3, \ldots, 10$. The value of $P(11)$ can be written as $-\frac{m}{n}$, where m and n are relatively prime positive integers. Determine $m + n$.

14. 3 Elm trees, 4 Dogwood trees, and 5 Oak trees are to be planted in a line in front of a library such that

i) No two Elm trees are next to each other.
ii) No Dogwood tree is adjacent to an Oak tree.
iii) All of the trees are planted.

How many ways can the trees be situated in this manner?

15. In triangle ABC, we have $BC = 13, CA = 37$, and $AB = 40$. Points D, E, and F are selected on BC, CA, and AB respectively such that AD, BE, and CF concur at the circumcenter of ABC. The value of

$$\frac{1}{AD} + \frac{1}{BE} + \frac{1}{CF}$$

can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Determine $m+n$.