Mock AIME 3

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1. Three circles are mutually externally tangent. Two of the circles have radii 3 and 7. If the area of the triangle formed by connecting their centers is 84, then the area of the third circle is $k\pi$ for some integer k. Determine k.

2. Let N denote the number of 7 digit positive integers have the property that their digits are in increasing order. Determine the remainder obtained when N is divided by 1000. (Repeated digits are allowed.)

3. A function $f(x)$ is defined for all real numbers x. For all non-zero values x, we have

$$2f(x) + f\left(\frac{1}{x}\right) = 5x + 4$$

Let S denote the sum of all of the values of x for which $f(x) = 2004$. Compute the integer nearest to S.

4. ζ_1, ζ_2, and ζ_3 are complex numbers such that

$$\begin{aligned} \zeta_1 + \zeta_2 + \zeta_3 &= 1 \\ {\zeta_1}^2 + {\zeta_2}^2 + {\zeta_3}^2 &= 3 \\ {\zeta_1}^3 + {\zeta_2}^3 + {\zeta_3}^3 &= 7 \end{aligned}$$

Compute ${\zeta_1}^7 + {\zeta_2}^7 + {\zeta_3}^7$.

5. In Zuminglish, all words consist only of the letters M, O, and P. As in English, O is said to be a vowel and M and P are consonants. A string of M's, O's, and P's is a word in Zuminglish if and only if between any two O's there appear at least two consonants. Let N denote the number of 10-letter Zuminglish words. Determine the remainder obtained when N is divided by 1000.

6. Let S denote the value of the sum

$$\sum_{n=1}^{9800} \frac{1}{\sqrt{n + \sqrt{n^2 - 1}}}$$

S can be expressed as $p + q\sqrt{r}$, where p, q, and r are positive integers and r is not divisible by the square of any prime. Determine $p + q + r$.

7. $ABCD$ is a cyclic quadrilateral that has an inscribed circle. The diagonals of $ABCD$ intersect at P. If $AB = 1$, $CD = 4$, and $BP : DP = 3 : 8$, then the area of the inscribed circle of $ABCD$ can be expressed as $\frac{p\pi}{q}$, where p and q are relatively prime positive integers. Determine $p + q$.

8. Let N denote the number of 8-tuples $(a_1, a_2, \dots, a_8)$ of real numbers such that $a_1 = 10$ and

$$\begin{aligned} \left|{a_1}^2 - {a_2}^2\right| &= 10 \\ \left|{a_2}^2 - {a_3}^2\right| &= 20 \\ &\cdots \\ \left|{a_7}^2 - {a_8}^2\right| &= 70 \\ \left|{a_8}^2 - {a_1}^2\right| &= 80 \end{aligned}$$

Determine the remainder obtained when N is divided by 1000.

9. ABC is an isosceles triangle with base $\overline{AB}$. D is a point on $\overline{AC}$ and E is the point on the extension of $\overline{BD}$ past D such that $\angle BAE$ is right. If $BD = 15, DE = 2$, and $BC = 16$, then CD can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Determine $m + n$.

10. $\{A_n\}_{n\geq 1}$ is a sequence of positive integers such that

$$a_n = 2a_{n-1} + n^2$$

for all integers $n > 1$. Compute the remainder obtained when a_{2004} is divided by 1000 if $a_1 = 1$.

11. ABC is an acute triangle with perimeter 60. D is a point on $\overline{BC}$. The circumcircles of triangles ABD and ADC intersect $\overline{AC}$ and $\overline{AB}$ at E and F respectively such that $DE = 8$ and $DF = 7$. If $\angle EBC \cong \angle BCF$, then the value of $\frac{AE}{AF}$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

12. Determine the number of integers n such that $1 \leq n \leq 1000$ and $n^{12} - 1$ is divisible by 73.

13. Let S denote the value of the sum

$$\left(\frac{2}{3}\right)^{2005} \cdot \sum_{k=1}^{2005} \frac{k^2}{2^k} \cdot \binom{2005}{k}$$

Determine the remainder obtained when S is divided by 1000.

14. Circles ω_1 and ω_2 are centered on opposite sides of line l, and are both tangent to l at P. ω_3 passes through P, intersecting l again at Q. Let A and B be the intersections of ω_1 and ω_3, and ω_2 and ω_3 respectively. AP and BP are extended past P and intersect ω_2 and ω_1 at C and D respectively. If $AD = 3, AP = 6, DP = 4$, and $PQ = 32$, then the area of triangle PBC can be expressed as $\frac{p\sqrt{q}}{r}$, where p, q, and r are positive integers such that p and r are coprime and q is not divisible by the square of any prime. Determine $p + q + r$.

15. Let Ω denote the value of the sum

$$\sum_{k=1}^{40} \cos^{-1}\left(\frac{k^2+k+1}{\sqrt{k^4+2k^3+3k^2+2k+2}}\right)$$

The value of $\tan(\Omega)$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.