Solutions to Mock AIME 3

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1. Answer: **196**. Let r denote the radius of the third circle. Then the sides of the triangle are $10, 3+r$, and $7+r$. Using Heron's formula and equating this with the given area, we have $84 = \sqrt{(10+r)(r)(7)(3)}$ from which $r = 14, -24$. Since r is positive, it follows that the area of the third circle is 196π.

2. Answer: **435**. Consider placing 7 X's and 8 +'s into a 15 character string. Replace each X with 1 plus the number of +'s that appear to its left. It follows that the result is a 7 digit number with its digits in increasing order. This bijection establishes that the number N is $\binom{15}{7} = 1435$, and the answer follows.

3. Answer: **601**. Substituting $\frac{1}{x}$ produces $2f\left(\frac{1}{x}\right) + f(x) = \frac{5}{x} + 4$. Subtract this from twice the given to obtain $3f(x) = 10x + 4 - \frac{5}{x}$. The non-zero solutions to $f(x) = 2004$ are therefore solutions to $10x^2 - 6008x - 5 = 0$. It follows that their sum is 600.8. (We don't care about $f(0)$ since if $f(0) = 2004$, the value of S is unchanged.) 601 is the nearest integer.

4. Answer: **071**. Consider the following algebra:

$$\begin{aligned}
1 = (\zeta_1 + \zeta_2 + \zeta_3)^2 &= \left({\zeta_1}^2 + {\zeta_2}^2 + {\zeta_3}^2\right) + 2(\zeta_1\zeta_2 + \zeta_2\zeta_3 + \zeta_3\zeta_1) \\
&= 3 + 2(\zeta_1\zeta_2 + \zeta_2\zeta_3 + \zeta_3\zeta_1) \\
\Longrightarrow \zeta_1\zeta_2 + \zeta_2\zeta_3 + \zeta_3\zeta_1 &= -1 \\
3 = \left({\zeta_1}^2 + {\zeta_2}^2 + {\zeta_3}^2\right)(\zeta_1 + \zeta_2 + \zeta_3) &= \left({\zeta_1}^3 + {\zeta_2}^3 + {\zeta_3}^3\right) + \sum_{Sym} {\zeta_1}^2\zeta_2 \\
&= 7 + \sum_{Sym} {\zeta_1}^2\zeta_2 \\
\Longrightarrow \sum_{Sym} {\zeta_1}^2\zeta_2 &= -4 \\
1 = (\zeta_1 + \zeta_2 + \zeta_3)^3 &= \left({\zeta_1}^3 + {\zeta_2}^3 + {\zeta_3}^3\right) + 3\sum_{Sym} {\zeta_1}^2\zeta_2 + 6\zeta_1\zeta_2\zeta_3 \\
&= 7 - 12 + 6\zeta_1\zeta_2\zeta_3 \\
\Longrightarrow \zeta_1\zeta_2\zeta_3 &= 1
\end{aligned}$$

But now, $(x - \zeta_1)(x - \zeta_2)(x - \zeta_3) = x^3 - (x^2 + x + 1)$. If we write $S_n = {\zeta_1}^n + {\zeta_2}^n + {\zeta_3}^n$, we have $S_{n+3} = S_{n+2} + S_{n+1} + S_n$. With this recursion, we find $S_4 = 7 + 3 + 1 = 11$, $S_5 = 11 + 7 + 3 = 21$, $S_6 = 21 + 11 + 7 = 39$, and $S_7 = 39 + 21 + 11 = 71$.

5. Answer: **936**. A ten letter word in Zuminglish contains between 0 and 4 (inclusive) O's. If there are no O's, each letter is either M or P. There are 1024 such words. If there is one O, then we have 10 places to insert O and 9 choices between M and P. There are $10 \cdot 512 = 5120$ such words. If there are $2 \leq k \leq 4$ O's, then the k O's partition the word into $k+1$ blocks (some possibly empty) of M's and P's. Since between any two O's there must be at least two letters, we have only $10 - k - 2 \cdot (k-1) = 12 - 3k$ consonants to insert by choice into $k+1$ slots. This can be accomplished in $\binom{12-2k}{k}$ ways. Since k O's corresponds to $10 - l$ choices between M and P, for $k = 2, 3, 4$ we have $\binom{8}{2} \cdot 2^8 = 7168$, $\binom{6}{3} \cdot 2^7 = 2560$, and $\binom{4}{4} \cdot 2^6 = 64$ different Zuminglish words with k O's repsectively. Adding, there are $1024 + 5120 + 7168 + 2560 + 64 = 15936$ such Zuminglish words.

6. Answer: **121**. The key lies in noticing that $\sqrt{n + \sqrt{n^2-1}} = \frac{1}{\sqrt{2}} \cdot \sqrt{2n + 2\sqrt{n^2-1}} = \frac{1}{\sqrt{2}} \cdot \left(\sqrt{n+1} + \sqrt{n-1}\right)$, since we may now write

$$\begin{aligned}
\sum_{n=1}^{9800} \frac{1}{\sqrt{n + \sqrt{n^2-1}}} &= \sqrt{2} \sum_{n=1}^{9800} \frac{1}{\sqrt{n+1} + \sqrt{n-1}} \\
&= \frac{1}{\sqrt{2}} \sum_{n=1}^{9800} \sqrt{n+1} - \sqrt{n-1} \\
&= \frac{1}{\sqrt{2}} \cdot \left(\sqrt{9801} + \sqrt{9800} - \sqrt{1} - \sqrt{0}\right) \\
&= 70 + 49\sqrt{2}
\end{aligned}$$

And it follows that the answer is $70 + 49 + 2 = 121$.

7. Answer: **049**. Because $ABCD$ has an incircle, $AD + BC = AB + CD = 5$. Suppose that $AD : BC = 1 : \gamma$. Then $3 : 8 = BP : DP = (AB \cdot BC) : (CD \cdot DA) = \gamma : 4$. We obtain $\gamma = \frac{3}{2}$, which substituted into $AD + BC = 5$ gives $AD = 2$, $BC = 3$. Now, the area of $ABCD$ can be obtained via Brahmagupta's formula: $s = \frac{1+2+3+4}{2} = 5$, $K = \sqrt{(s-a)(s-b)(s-c)(s-d)} = \sqrt{24}$ and $K = rs = 5r$, where r is the inradius of $ABCD$. Thus, $r = \frac{\sqrt{24}}{5}$ from which its area $\frac{24\pi}{25}$ yields the answer $24 + 25 = 49$.

8. **472**. Removing the absolute value signs, we have $a_1^2 = \pm 10 + a_2^2 = \pm 10 \pm 20 + a_3^2 = \cdots = \pm 10 \pm 20 \pm 30 \pm \cdots \pm 80 + a_1^2$. We see the constraint that the signs of each $\pm$ must be chosen such that $100 \pm 10 \pm \cdots$ is non-negative at each step, since at each step it corresponds to the value of the square of some a_i, in addition to the sum $\pm 10 \pm 20 \pm \cdots \pm 80$ being 0. Dividing out a factor of 10, we are interested in the subsets of $\{1, 2, 3, \ldots, 8\}$ with a sum of 18. Each subset corresponds to a zero-sum choice of signs in $10 \pm 1 \pm \cdots \pm n$ for $n = 1, 2, \cdots, 8$. The number of 8-tuples that correspond to each of these subsets is 0 if for any n the partial sum is negative, otherwise 2^k, where k is the number of values of $1 \leq n < 8$ for which $10 \pm 1 \pm \cdots \pm n$ is positive. This follows from recalling that each partial sum is a tenth of the square of some a_i, which allows for a choice of signs in each value except in a_1.

   The subset $\{3, 7, 8\}$ corresponds to $10 - 1 - 2 + 3 - 4 - 5 - 6 + 7 + 8 = 10$, but the partial sum is negative for $n = 6$, which corresponds to $a_7^2 = -10 - 20 + 30 - 40 - 50 - 60 + a_1^2 = -10$. The subset $\{4, 6, 8\}$, however, corresponds to $10 - 1 - 2 - 3 + 4 - 5 + 6 - 7 + 8 = 10$, which is positive for all of its partial sums. Thus, there are 128 ordered 8-tuples corresponding to this subset. Continuing in this fashion:

$$\begin{array}{rcl rcl rcl} \{5,6,7\} & \rightarrow & 64. & \{1,2,7,8\} & \rightarrow & 0. & \{1,3,6,8\} & \rightarrow & 128. \\ \{1,4,5,8\} & \rightarrow & 128. & \{2,3,5,8\} & \rightarrow & 128. & \{1,4,6,7\} & \rightarrow & 128. \\ \{2,3,6,7\} & \rightarrow & 128. & \{2,4,5,7\} & \rightarrow & 128. & \{3,4,5,6\} & \rightarrow & 128. \\ \{1,2,3,4,8\} & \rightarrow & 128. & \{1,2,3,5,7\} & \rightarrow & 128. & \{1,2,4,5,6\} & \rightarrow & 128. \end{array}$$

Where the subset $\{5,6,7\}$ is the only one for which we encounter an $a_i^2 = 0$, which halves the number of corresponding 8-tuples. Adding these numbers, we find that there are 1472 such 8-tuples, whence the answer.

9. Answer: **225**. Draw in altitude $\overline{CF}$ and denote its intersection with $\overline{BD}$ by P. Since ABC is isosceles, $AF = FB$. Now, since BAE and BFP are similar with a scale factor of 2, we have $BP = \frac{1}{2}BE = \frac{17}{2}$, which also yields $PD = BD - BP = 15 - \frac{17}{2} = \frac{13}{2}$. Now, applying Menelaus to triangle ADB and collinear points C, P, and F, we obtain

$$\begin{aligned} \frac{AC}{CD}\frac{DP}{PB}\frac{BF}{FA} &= \frac{AC}{CD}\frac{DP}{PB} = -1 \\ \Longrightarrow |CD| = AC \cdot \frac{DP}{PB} &= 16 \cdot \frac{\left(\frac{13}{2}\right)}{\left(\frac{17}{2}\right)} = \frac{208}{17} \end{aligned}$$

where the minus sign was a consequence of directed distances.[1] The answer is therefore $208 + 17 = 225$.

10. Answer: **058**. $a_{2004} = 2a_{2003} + 2004^2 = 2\left(2a_{2002} + 2003^2\right) + 2004^2 = \cdots = 2^{2003} \cdot 1^2 + 2^{2002} \cdot 2^2 + 2^{2001} \cdot 3^2 + \cdots + 2^0 \cdot 2004^2$. We subtract a_{2004} from twice itself two times to telescope this sum:

$$\begin{aligned} a_{2004} = 2a_{2004} - a_{2004} &= \left(2^{2004} \cdot 1 + 2^{2003} \cdot 4 + \cdots + 2 \cdot 2004^2\right) - \left(2^{2003} \cdot 1 + \cdots + 2^0 \cdot 2004^2\right) \\ &= 2^{2004} + 3 \cdot 2^{2003} + \cdots + 4007 \cdot 2^1 - 2004^2 \\ a_{2004} = 2a_{2004} - a_{2004} &= \left(2^{2005} + 3 \cdot 2^{2004} + \cdots + 4007 \cdot 2^2 - 2 \cdot 2004^2\right) \\ &\quad - \left(1 \cdot 2^{2004} + 3 \cdot 2^{2003} + \cdots + 4007 \cdot 2^1 - 2004^2\right) \\ &= 2^{2005} + 2 \cdot \left(2^{2004} + 2^{2003} + \cdots + 2^2\right) - 2 \cdot 4007 - 2004^2 \\ a_{2004} &\equiv 2^5 + 2\left(2^{2005} - 4\right) - 14 - 16 \pmod{1000} \\ &\equiv 32 + 64 - 8 - 30 = 58 \pmod{1000} \end{aligned}$$

11. Answer: **035**. Since $BDEA$ is cyclic, $\angle EBD \cong \angle EAD$. Similarly, $\angle DCF \cong \angle DAF$. Since we are given $\angle BCF \cong \angle EBC$, we have $\angle DAB \cong \angle CAD$. Because $\overline{CD}$ and $\overline{DF}$ are intercepted by congruent angles in the same circle, $DF = CD = 7$. Similarly, $DB = 8$. Now, by the angle bisector theorem, $AC = 7x$ and $AB = 8x$. Since the perimeter of ABC is 60, $15 + 15x = 60$ and $x = 3$, so that $AC = 21$ and $AB = 24$. Now, by Power of a Point from B, $BF = \frac{BD \cdot BC}{BA} = \frac{8 \cdot 15}{24} = 5$ and $CE = \frac{CD \cdot CB}{CA} = \frac{7 \cdot 15}{21} = 5$. Subtracting these lengths from AB and AC respectively, we find that $AF = 19$ and $AE = 16$. It follows that the answer is $16 + 19 = 35$.

[1] A system of linear measure in which for any points A and B, $AB = -BA$.

12. Answer: **164**. We are interested in values of n for which $n^{12}-1=(n^6+1)(n^6-1)=(n^2+1)(n^4-n^2+1)(n+1)(n-1)(n^4+n^2+1)$ is divisible by 73. We note that 73 is prime, so that at most two distinct residues satisfy $r^2 \equiv k \pmod{73}$. First, we need to solve for the square roots of -1, since they are the zeros of $n^2+1 \equiv 0 \pmod{73}$. If we do not notice that $27^2 = 729 \equiv -1 \pmod{73}$, then we can work from $3^2+8^2=73$, since $8^2 \equiv -(3^2) \pmod{73} \iff \left(8\cdot 3^{-1}\right)^2 \equiv -1 \pmod{73}$. 3^{-1} is the unique residue r such that $3r \equiv 1 \pmod{73}$. It is the only integer in the set $\{\frac{1}{3}, \frac{74}{3}, \frac{147}{3}\}$, or 49. Thus, $8\cdot 49 \equiv 27 \pmod{73}$ is a square root of -1. The other is $73-27=46$. (We will omit writing $\pmod{73}$ understanding that all of the following algebra is in this numeric system.)

In solving $n^2-n+1=n(n-1)+1\equiv 0$, if we do not notice that $8\cdot 9=72\equiv -1$, we can proceed by completing the square. Since $n^2-n+1\equiv n^2-74n+1$, $(n-37)^2 \equiv 1368 \equiv 1295 \equiv \cdots \equiv 784$. We stop at 784 because it is the square of 28. Thus, $n-37\equiv \pm 28$, or $n\equiv 9,65$. The solutions to $n^2+n+1=(n+1)\cdot n+1\equiv 0$ are now easy to find since we are merely substituting $n+1$ in place of n. Thus, $n^2+n+1\equiv 0 \iff n\equiv 8,64$.

To solve $n^4-n^2+1=n^2(n^2-1)+1\equiv 0$, we take the square roots of the solutions to $n(n-1)+1\equiv 0$, since $n^2 \equiv 9,65$. For $n^9 \equiv 9$, we have $n\equiv 3,70$. The solutions to $n^2\equiv 65\equiv -81\equiv (27\cdot 9)^2 \equiv 24^2$ are $n\equiv 24,49$.

In solving $n^4+n^2+1\equiv 0$, we play the same trick once more, noting that the solutions to $n^2\equiv 64$ are $n\equiv 8,65$. The solutions to $n^2\equiv 8\equiv 81$ are $n\equiv 9,64$.

Combining these with the trivial solutions, we have $n\equiv 1,3,8,9,24,27,46,49,64,65,70,72 \pmod{73}$. Now, $1000=13\cdot 73+51$, so we count all 12 of these solutions 13 times, and the 8 residues less than or equal to 51 one more time. This gives a total of $13\cdot 12+8=164$ solutions.

13. Answer: **115**. We note the combinatorial identity $k\binom{n}{k}=n\binom{n-1}{k-1}$ and write

$$k^2\binom{2005}{k} = 2005k\binom{2004}{k-1} = 2005\left((k-1)\binom{2004}{k-1}+\binom{2004}{k-1}\right) = 2005\left(2004\binom{2003}{k-2}+\binom{2004}{k-1}\right)$$

Employing this result,

$$\begin{aligned}
S &= \left(\frac{2}{3}\right)^{2005}\cdot\sum_{k=1}^{2005}\frac{k^2}{2^k}\binom{2005}{k} \\
&= \left(\frac{2}{3}\right)^{2005}\cdot 2005\sum_{k=1}^{2005}\frac{1}{2^k}\left(2004\binom{2003}{k-2}+\binom{2004}{k-1}\right) \\
&= \left(\frac{2}{3}\right)^{2005}\cdot 2005\left(501\cdot\sum_{k=0}^{2003}\frac{1}{2^k}\binom{2003}{k}+\frac{1}{2}\sum_{k=0}^{2004}\frac{1}{2^k}\binom{2004}{k}\right) \\
&= 2005\left(501\cdot\frac{4}{9}\cdot\sum_{k=0}^{2003}\frac{2^{2003-k}}{3^{2003-k}}\frac{1^k}{3^k}\binom{2003}{k}+\frac{1}{2}\cdot\frac{2}{3}\cdot\sum_{k=0}^{2004}\frac{2^{2004-k}}{3^{2004-k}}\frac{1^k}{3^k}\binom{2004}{k}\right) \\
&= 2005\left(\frac{668}{3}\cdot\left(\frac{2}{3}+\frac{1}{3}\right)^{2003}+\frac{1}{3}\cdot\left(\frac{2}{3}+\frac{1}{3}\right)^{2004}\right) = 447115.
\end{aligned}$$

14. Answer: **468**. We invert about P with radius 1, mapping the circles ω_1 and ω_2 to lines ω_1' and ω_2', each parallel to l, and ω_3 to a line ω_3' that intersects ω_1' and ω_2' at A' and B' respectively. Q' is the intersection of l and ω_3', and C' and D' are the intersections of the extensions of $A'P$ and $B'P$ past P to ω_2' and ω_1' respectively.

We have $PQ' = \frac{1}{32}, PA' = \frac{1}{6}$, and $PD' = \frac{1}{4}$. The inversive distance formula gives $A'D' = \frac{R^2 \cdot AD}{AP \cdot DP} = \frac{1}{8}$. The crossed ladders theorem asserts

$$\frac{1}{A'D'} + \frac{1}{B'C'} = \frac{1}{PQ'}$$

from which $B'C' = \frac{1}{24}$. However, it is clear in the inverted figure that triangles $C'B'P$ and $A'D'P$ are similar. Therefore, $PC' = \frac{1}{18}$ and $PB' = \frac{1}{12}$.

But inversion is its own inverse transformation. Hence, $PC = 18$ and $PB = 12$. The inversive distance formula gives $BC = \frac{R^2 \cdot B'C'}{PB' \cdot PC'} = \frac{18 \cdot 12}{24} = 9$. Finally, the area of PBC may be found via Heron's formula: $K = \sqrt{\frac{39}{2}\frac{21}{2}\frac{15}{2}\frac{3}{2}} = \frac{9\sqrt{455}}{4}$. The answer is therefore $455 + 9 + 4 = 468$.

15. Answer: **041**. The cosine inverse subtraction formula,

$$\cos^{-1}(a) - \cos^{-1}(b) = \cos^{-1}\left(ab + \sqrt{1-a^2}\sqrt{1-b^2}\right)$$

will be the vehicle for telescoping this sum. It can be shown via AM-GM that $x^4 + 2x^3 + 3x^2 + 2x + 2$ has no real roots, so we inspect for imaginary solutions among Gaussian integers. Finding that $x = \pm i$ are solutions, we factor accordingly: $x^4 + 2x^3 + 3x^2 + 2x + 2 = (x^2 + 1)(x^2 + 2x + 2) = (x^2 + 1)((x + 1)^2 + 1)$ and note that the other two roots are $-1 \pm i$. If this sum is going to telescope, it ought to be due to

$$\cos^{-1}\left(\frac{k^2+k+1}{\sqrt{k^4+2k^3+3k^2+2k+2}}\right) = \cos^{-1}\left(f\left((k+1)^2+1\right)\right) - \cos^{-1}\left(f(k^2+1)\right)$$

for some function f. Because there is a square root in the denominator on the left, we conjecture that $f(x) = \frac{1}{\sqrt{x}}$. Checking this reveals remarkable simplification:

$$\begin{aligned}
&\cos^{-1}\left(\frac{1}{\sqrt{(k+1)^2+1}}\right) - \cos^{-1}\left(\frac{1}{\sqrt{k^2+1}}\right) \\
= \;&\cos^{-1}\left(\frac{1}{\sqrt{k^2+1}\sqrt{(k+1)^2+1}} + \sqrt{1-\frac{1}{k^2+1}}\sqrt{1-\frac{1}{(k+1)^2+1}}\right) \\
= \;&\cos^{-1}\left(\frac{1}{\sqrt{k^4+2k^3+3k^2+2k+2}} + \frac{k(k+1)}{\sqrt{k^2+1}\sqrt{(k+1)^2+1}}\right) \\
= \;&\cos^{-1}\left(\frac{k^2+k+1}{\sqrt{k^4+2k^3+3k^2+2k+2}}\right)
\end{aligned}$$

Hence, $\Omega = \cos^{-1}\left(\frac{1}{\sqrt{41^2+1}}\right) - \cos^{-1}\left(\frac{1}{\sqrt{1^2+1}}\right) = \cos^{-1}\left(\frac{1+41}{\sqrt{2\cdot(41^2+1)}}\right) = \cos^{-1}\left(\frac{42}{\sqrt{2\cdot 1682}}\right) = \cos^{-1}\left(\frac{21}{29}\right)$. Recalling that a triangle of sides 20, 21, and 29 is a right triangle, it follows that $\tan(\Omega) = \frac{20}{21}$, whence the answer $20 + 21 = 41$.