Mock AIME 4

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1. For how many positive integers $n > 1$ is it possible to express 2005 as the sum of n distinct positive integers?

2. $a_1, a_2, \ldots$ is a sequence of real numbers where a_n is the arithmetic mean of the previous $n-1$ terms for $n > 3$ and $a_{2004} = 7$. $b_1, b_2, \ldots$ is a sequence of real numbers in which b_n is the geometric mean of the previous $n-1$ terms for $n > 3$ and $b_{2005} = 6$. If $a_i = b_i$ for $i = 1, 2, 3$ and $a_1 = 3$, then compute the value of $a_2^2 + a_3^2$.

3. Compute the largest integer n such that $2005^{2^{100}} - 2003^{2^{100}}$ is divisible by 2^n.

4. $ABCDEFG$ is a regular heptagon, and P is a point in its interior such that ABP is equilateral. There exists a unique pair $\{m, n\}$ of relatively prime positive integers such that $m\angle CPE = \left(\frac{m}{n}\right)^\circ$. Compute the value of $m + n$.

5. Compute, to the nearest integer, the area of the region enclosed by the graph of $13x^2 - 20xy + 52y^2 - 10x + 52y = 563$.

6. Determine the remainder obtained when 1000! is divided by 2003.

7. $\mathcal{P}$ is a pyramid consisting of a square base and four slanted triangular faces such that all of its edges are equal in length. $\mathcal{C}$ is a cube of edge length 6. Six pyramids similar to $\mathcal{P}$ are constructed by taking points P_i (all outside of $\mathcal{C}$) where $i = 1, 2, \ldots, 6$ and using the nearest face of $\mathcal{C}$ as the base of each pyramid exactly once. The volume of the octahedron formed by the P_i (taking the convex hull) can be expressed as $m + n\sqrt{p}$ for some positive integers m, n, and p, where p is not divisible by the square of any prime. Determine the value of $m + n + p$.

8. A single atom of Uranium-238 rests at the origin. Each second, the particle has a 1/4 chance of moving one unit in the negative x direction and a 1/2 chance of moving in the positive x direction. If the particle reaches $(-3, 0)$, it ignites a fission that will consume the earth. If it reaches $(7, 0)$, it is harmlessly diffused. The probability that, eventually, the particle is safely contained can be expressed as $\frac{m}{n}$ for some relatively prime positive integers m and n. Determine the remainder obtained when $m + n$ is divided by 1000.

9. The value of the sum
$$\sum_{n=1}^{\infty} \frac{(7n+32) \cdot 3^n}{n \cdot (n+2) \cdot 4^n}$$
can be expressed in the form $\frac{p}{q}$, for some relatively prime positive integers p and q. Compute the value of $p + q$.

10. 100 blocks are selected from a crate containing 33 blocks of each of the following dimensions: $13 \times 17 \times 21$, $13 \times 17 \times 37$, $13 \times 21 \times 37$, and $17 \times 21 \times 37$. The chosen blocks are stacked on top of each other (one per cross section) forming a tower of height h. Compute the number of possible values of h.

11. 10 lines and 10 circles divide the plane into at most n disjoint regions. Compute n.

12. Determine the number of permutations of $1, 2, 3, 4, \ldots, 32$ such that if m divides n, the mth number divides the nth number.

13. x, y, and z are distinct non-zero integers such that $-7 \leq x, y, z \leq 7$. Compute the number of solutions (x, y, z) to the equation

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{x+y+z}$$

14. In triangle ABC, $BC = 27, CA = 32$, and $AB = 35$. P is the unique point such that the perimeters of triangles BPC, CPA, and APB are equal. The value of $AP + BP + CP$ can be expressed as $\frac{p+q\sqrt{r}}{s}$, where p, q, r, and s are positive integers such that there is no prime divisor common to p, q, and s, and r is not divisible by the square of any prime. Determine the value of $p + q + r + s$.

15. $ABCD$ is a convex quadrilateral in which $\overline{AB} \parallel \overline{CD}$. Let U denote the intersection of the extensions of $\overline{AD}$ and $\overline{BC}$. Ω_1 is the circle tangent to line segment $\overline{BC}$ which also passes through A and D, and Ω_2 is the circle tangent to $\overline{AD}$ which passes through B and C. Call the points of tangency M and S. Let O and P be the points of intersection between Ω_1 and Ω_2. Finally, $\overline{MS}$ intersects $\overline{OP}$ at V. If $AB = 2$, $BC = 2005$, $CD = 4$, and $DA = 2004$, then the value of UV^2 is some integer n. Determine the remainder obtained when n is divided by 1000.