Solutions to Mock AIME 4

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1. For how many positive integers $n > 1$ is it possible to express 2005 as the sum of n distinct positive integers?

   Answer: **061**. The sum of n distinct positive integers is at least $1+2+3+\cdots+n = \frac{n(n+1)}{2}$, but because we can exchange n with $n+k$ for any integer $k \geq 0$, the sum of n distinct positive integers can be *any* integer at least $\frac{n(n+1)}{2}$. We have $1+2+\cdots+62 = \frac{62\cdot 63}{2} = 1953$ and $1+2+\cdots+63 = \frac{63\cdot 64}{2} = 2016 > 2005$. Therefore, 2005 can be expressed as the sum of $n > 1$ integers for $n = 2, 3, \ldots, 61, 62$; 61 distinct values.

2. $a_1, a_2, \ldots$ is a sequence of real numbers where a_n is the arithmetic mean of the previous $n-1$ terms for $n > 3$ and $a_{2004} = 7$. $b_1, b_2, \ldots$ is a sequence of real numbers in which b_n is the geometric mean of the previous $n-1$ terms for $n > 3$ and $b_{2005} = 6$. If $a_i = b_i$ for $i = 1, 2, 3$ and $a_1 = 3$, then compute the value of $a_2^2 + a_3^2$.

   Answer: **180**. Note that $a_{2004} = \frac{a_1+a_2+a_3+\frac{a_1+a_2+a_3}{3}+\cdots}{2004}$ is symmetric with respect to a_1, a_2, and a_3. Therefore, it is the arithmetic mean of the first three terms. This implies that $a_2 + a_3 = 18$. By similar reasoning, b_{2005} is the geometric mean of b_1, b_2, and b_3, from which $b_2 b_3 = 72$. Since $b_2 b_3 = a_2 a_3$, we have $a_2^2 + a_3^2 = (a_2+a_3)^2 - 2a_2a_3 = 18^2 - 2\cdot 72 = 180$.

3. Compute the largest integer n such that $2005^{2^{100}} - 2003^{2^{100}}$ is divisible by 2^n.

   Answer: **103**. The expression factors as

   $$\left(2005^{2^{99}} + 2003^{2^{99}}\right)\cdots\left(2005^{2^k} + 2003^{2^k}\right)\cdots\left(2005^{2^0} + 2003^{2^0}\right)\left(2005^{2^0} - 2003^{2^0}\right)$$

   Each term is even, but since all odd squares are equivalent to 1 modulo 4, the only term that contains more than one factor of 2 is $2005^{2^0} + 2003^{2^0} = 4008 = 8\cdot 501$. Thus, treating each of the 101 terms as once divisible by 2 undercounts the number of factors by 2, giving an answer of 103.

4. $ABCDEFG$ is a regular heptagon, and P is a point in its interior such that ABP is equilateral. There exists a unique pair $\{m, n\}$ of relatively prime positive integers such that $m\angle CPE = \left(\frac{m}{n}\right)^\circ$. Compute the value of $m+n$.

   Answer: **667**. Since ABP is equilateral, $BP = BA = BC$, hence $\angle BCP \cong \angle CPB$. Let α denote the degree measure of each of the angles of $ABCDEFG$. Then $m\angle PCB = \alpha - 60^\circ$

from which $m\angle CPB = m\angle BCP = 120^\circ - \frac{\alpha}{2}$ and $m\angle PCD = \frac{3\alpha}{2} - 120^\circ$. By symmetry, P lies on the angle bisector of $\angle DEF$, thus $m\angle DEP = \frac{\alpha}{2}$. Finally, as $m\angle CDE = \alpha$, we have $m\angle EPC = 360^\circ - \frac{\alpha}{2} - \alpha - \left(\frac{3\alpha}{2} - 120^\circ\right) = 480^\circ - 3\alpha$. Computing $\alpha = 180^\circ - \frac{360^\circ}{7} = \frac{900^\circ}{7}$, we find that $m\angle EPC = \frac{660^\circ}{7}$.

5. Compute, to the nearest integer, the area of the region enclosed by the graph of $13x^2 - 20xy + 52y^2 - 10x + 52y = 563$.

   Answer: **075**. We apply the algebra

$$\begin{aligned}
13x^2 + \left[52\left(y+\frac{1}{2}\right)^2 - 13\right] - 20x\left(y+\frac{1}{2}\right) &= 563\\
13\left(\frac{x}{2}\right)^2 + 13\left(y+\frac{1}{2}\right)^2 - 10\left(\frac{x}{2}\right)\left(y+\frac{1}{2}\right) &= 144\\
13x_0^2 + 13y_0^2 - 10x_0y_0 &= 144
\end{aligned}$$

   where $x_0 = \frac{x}{2}$ and $y_0 = y + \frac{1}{2}$. It is clear that the transformation $(x, y) \to (x_0, y_0)$ shifts (x, y) up half of a unit and then scales this image by a factor of $\frac{1}{2}$ along the x direction. Therefore, the area of the region enclosed by the original equation is twice that enclosed by the new region.
   Setting $x_0 = y_0$, we find the points $\pm(3, 3)$. Setting $x_0 = -y_0$, we find the points $\pm(-2, 2)$. The new graph encloses an ellipse that has a semimajor-axis length of $3\sqrt{2}$ and semiminor-axis length of $2\sqrt{2}$. Its area is then $\left(2\sqrt{2}\right)\left(3\sqrt{2}\right)\pi = 12\pi$. Therefore, the original graph encloses an ellipse of area $24\pi = 75.398\ldots$

6. Determine the remainder obtained when 1000! is divided by 2003.

   Answer: **002**. Note that 2003 is prime. Now, by Wilson's Theorem, $2002! \equiv -1 \pmod{2003}$, but $2002! \equiv (1 \cdot 2 \cdots 1001)(-1001 \cdots -1) \equiv -(1001!)^2 \equiv -1 \pmod{2003}$. Hence, $1001! \equiv 1 \pmod{2003}$ or $1001! \equiv 2002 \pmod{2003}$. Dividing by 1001, two possible answers are 2001 and 2 respectively. Because the answer must be less than 1000, it must be that the remainder is 2.

7. $\mathcal{P}$ is a pyramid consisting of a square base and four slanted triangular faces such that all of its edges are equal in length. $\mathcal{C}$ is a cube of edge length 6. Six pyramids similar to $\mathcal{P}$ are constructed by taking points P_i (all outside of $\mathcal{C}$) where $i = 1, 2, \ldots, 6$ and using the nearest face of $\mathcal{C}$ as the base of each pyramid exactly once. The volume of the octahedron formed by the P_i (taking the convex hull) can be expressed as $m + n\sqrt{p}$ for some positive integers m, n, and p, where p is not divisible by the square of any prime. Determine the value of $m+n+p$.

   Answer: **434**. By a Pythagoras argument, the height of each pyramid from P_i to the nearest face of $\mathcal{C}$ is $3\sqrt{2}$. Therefore, the distance from opposite vertices of the octohedron is $6 + 2 \cdot 3\sqrt{2} = 6 \cdot \left(1+\sqrt{2}\right)$. Let P_s and P_t be a pair of opposite vertices. The square formed by the other four vertices of the octahedron has area $\frac{1}{2} \cdot \left(6 \cdot \left(1+\sqrt{2}\right)\right)^2 = 18\left(3+2\sqrt{2}\right)$. Finally, the volume is given by $\frac{1}{3}\left(6(1+\sqrt{2})\right)\left(18(3+2\sqrt{2})\right) = 252 + 180\sqrt{2}$.

8. A single atom of Uranium-238 rests at the origin. Each second, the particle has a $1/4$ chance of moving one unit in the negative x direction and a $1/2$ chance of moving in the positive x direction. If the particle reaches $(-3,0)$, it ignites a fission that will consume the earth. If it reaches $(7,0)$, it is harmlessly diffused. The probability that, eventually, the particle is safely contained can be expressed as $\frac{m}{n}$ for some relatively prime positive integers m and n. Determine the remainder obtained when $m+n$ is divided by 1000.

Answer: **919**. Let $p(n)$ be the probability that the atom is safely contained if released from $(n,0)$. In this notation, $P(-3)=0, P(7)=1$. Now, since the particle is twice as likely to move right as it is likely to move left, $P(n)=\frac{2}{3}P(n+1)+\frac{1}{3}P(n-1)$ or equivalently $P(n+1)=\frac{3P(n)-P(n-1)}{2}$ for $-2\le n\le 6$. Let $P(-2)=r$. Then $P(-1)=\frac{3r}{2}, P(0)=\frac{7r}{4}, P(1)=\frac{15r}{8}, P(2)=\frac{31r}{16}, P(3)=\frac{63r}{32}, P(4)=\frac{127r}{64}, P(5)=\frac{255r}{128}$, and $P(6)=\frac{511r}{256}$. But now $\frac{511r}{256}=\frac{2}{3}+\frac{1}{3}\frac{255r}{128}$ which gives $r=\frac{512}{1023}$ from which $\frac{m}{n}=\frac{7}{4}\frac{512}{1023}=\frac{896}{1023}$.

9. The value of the sum

$$\sum_{n=1}^{\infty}\frac{(7n+32)\cdot 3^n}{n\cdot(n+2)\cdot 4^n}$$

can be expressed in the form $\frac{p}{q}$, for some relatively prime positive integers p and q. Compute the value of $p+q$.

Answer: **035**. Note that $\frac{7n+32}{n\cdot(n+2)}=\frac{16}{n}-\frac{9}{n+2}$ so that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(7n+32)}{n\cdot(n+2)}\frac{3^n}{4^n} &= \sum_{n=1}^{\infty}\frac{16}{n}\frac{3^n}{4^n}-\sum_{n=1}^{\infty}\frac{9}{n+2}\frac{3^n}{4^n}\\ &= \sum_{n=1}^{\infty}\frac{16}{n}\frac{3^n}{4^n}-\sum_{n=1}^{\infty}\frac{16}{n+2}\frac{3^{(n+2)}}{4^{(n+2)}}\\ &= \sum_{n=1}^{\infty}\frac{16}{n}\frac{3^n}{4^n}-\sum_{n=3}^{\infty}\frac{16}{n}\frac{3^n}{4^n}\\ &= \frac{16}{1}\frac{3}{4}+\frac{16}{2}\frac{9}{16}=\frac{33}{2}\end{aligned}$$

10. 100 blocks are selected from a crate containing 33 blocks of each of the following dimensions: $13\times17\times21$, $13\times17\times37$, $13\times21\times37$, and $17\times21\times37$. The chosen blocks are stacked on top of each other (one per cross section) forming a tower of height h. Compute the number of possible values of h.

Answer: **595**. Subtract 13 from each dimension and then divide everything by 4. This changes each height of h to a new height of $h'=\frac{h-1300}{4}$. This injective mapping guarantees that the number of possible h is equal to the number of possible h'. Now we are working with the sum $h'=x_1+x_2+\cdots+x_{100}$ where $x_i\in\{0,1,2,6\}$. Obviously, h' is bounded by $0\le s\le 600$. If we initially take $k\ge 97$ 6's and $97-k$ 0's, we have three more choices. With three choices from $\{0,1,2\}$, we can make any number in $\{0,1,2,3,4,5,6\}$, so all of the sums $0\le h'\le 97\cdot6+6=588$ are possible. For h' to exceed 588, we must take 98 6's. Two choices from $\{0,1,2,6\}$, we can make $\{0,1,2,3,4,6,7,8,12\}$, which, when added to 98 6's, give the values $h' \in \{588,589,590,591,592,594,595,596,600\}$. The fact that we are actually choosing 100 blocks from the crate disqualifies towers built with 100 of the same dimension since each dimension appears on 99 blocks. Although there are many towers that give $h' = 100$ and $h' = 200$, the towers giving $h' = 0$ and $h' = 600$ are uniquely built from 100 13's and 100 37's. Thus, the possible h' are $1 \leq h' \leq 592$, $h' = 594, h' = 595$, and $h' = 596$. A total of 595 values.

11. 10 lines and 10 circles divide the plane into at most n disjoint regions. Compute n.

    Answer: **346**. Any arrangement of 10 lines and 10 circles can be constructed in any order. Ten lines such that no two are parallel and no three have a common intersection divide the plane into $1 + (1 + 2 + 3 + \cdots + 10) = 56$ regions. Now, each new circle creates additional regions equal in number to the number of new points of intersection between itself and the other lines and circles (or 1 if it intersects no other objects, but this is clearly not maximal.) Thus, the assumption that the lines divide the plane into as many rregions as possible is valid. Furthermore, we know that the answer is given by $56 + V_{\text{max}}$, where V_{max} is the number of intersections between two shapes such that at least one is a circle. Since a circle can intersect a line in at most two places, there are at most $2 \cdot 10 \cdot 10 = 200$ circle-line intersections. A pair of circles intersect at no more than two points, so there are at most $2 \cdot \binom{10}{2} = 90$ circle-circle points of intersection. Therefore, the optimal configuration yields $56+200+90 = 346$ regions.

12. Determine the number of permutations of $1, 2, 3, 4, \ldots, 32$ such that if m divides n, the mth number divides the nth number.

    Answer: **240**. Let $\pi(m)$ denote the mth number of the permutation. Because there there are exactly as many instances where $\pi(i)|\pi(j)$ as there are $i|j$ for $i \neq j$, it must be that m divides n if and only if $\pi(m)$ divides $\pi(n)$. Therefore, $\pi(m)$ must divide exactly $\lfloor \frac{32}{m} \rfloor$ numbers in the set $\{1, 2, 3, \ldots, 32\}$. It follows that $\pi(m)$ must have as many factors as m. This, in turn, implies that the permutation π can only shuffle sets S with the property that for every x and y in S, the number of factors of x equals the number of factors of y and $\left\lfloor \frac{32}{x} \right\rfloor = \left\lfloor \frac{32}{y} \right\rfloor$. The only such sets containing more than one element are $\{11, 13\}$, $\{14, 15\}$, $\{17, 19, 23, 29, 31\}$, $\{18, 20, 28\}$, $\{21, 22, 26, 27\}$, and $\{24, 30\}$. All m not in these sets must have $\pi(m) = m$. With the exception of $11, 13, 22$, and 26, all of these numbers have the property that their proper divisors d must obey $\pi(d) = d$. It follows that the only permutable sets are $\{11, 13\}$, $\{22, 26\}$, and $\{17, 19, 23, 29, 31\}$. The first two are linked - either both pairs of numbers are swapped or neither is swapped; two possibilities. The primes can be arranged in any order; 120 possibilities. All of these permutations are easily seen to satisfy the constraints.

13. x, y, and z are distinct non-zero integers such that $-7 \leq x, y, z \leq 7$. Compute the number of solutions (x, y, z) to the equation

    $$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{x+y+z}$$

    Answer: **504**. Obviously, $x + y + z \neq 0$. We clear the denominators and simplify:

    $$yz(x+y+z) + zx(x+y+z) + xy(x+y+z) = xyz$$

$$\begin{aligned} x^2(y+z)+y^2(z+x)+z^2(x+y)+2xyz &= 0 \\ (x+y)(y+z)(z+x) &= 0 \end{aligned}$$

It must be that $x=-y$, $y=-z$, or $z=-x$. Because x, y, and z are distinct, these three cases do not overlap. The first case has 14 possible pairs (x,y) and 12 choices of z for each of these pairs. Therefore, there are $3 \cdot 14 \cdot 12 = 504$ possible (x,y,z).

14. In triangle ABC, $BC = 27, CA = 32$, and $AB = 35$. P is the unique point such that the perimeters of triangles BPC, CPA, and APB are equal. The value of $AP + BP + CP$ can be expressed as $\frac{p+q\sqrt{r}}{s}$, where p, q, r, and s are positive integers such that there is no prime divisor common to p, q, and s, and r is not divisible by the square of any prime. Determine the value of $p+q+r+s$.

Answer: **171**. Algebra implies that $AP = k+27$, $BP = k+32$, and $CP = k+35$ for some k. Let the points of tangency between the incircle of ABC and $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ be D, E, and F respectively. Circles ω_1, ω_2, and ω_3 of radii $s-27$, $s-32$, and $s-35$ ($s = 47$, the semiperimeter) and centered at A, B, and C respectively, are pairwise externally tangent at D, E, and F. Let Ω be the circle tangent to ω_1, ω_2, and ω_3, at points T_1, T_2, and T_3 respectively, such that the ω_i lie in the interior of Ω. Let ζ denote the radius of Ω. By the common tangency, lines T_1A, T_2B, and T_3C concur at the center of Ω. P is the center of Ω because if we let $k = \zeta - s$, we have $AP = \zeta - (s-27) = k+27$, $BP = k+32$, and $CP = k+35$. By the Descartes Circle Theorem,

$$\begin{aligned} 2\left(\frac{1}{12^2}+\frac{1}{15^2}+\frac{1}{20^2}+\frac{1}{\zeta^2}\right) &= \left(\frac{1}{12}+\frac{1}{15}+\frac{1}{20}-\frac{1}{\zeta}\right)^2 \\ 2\left(25+16+9+\frac{3600}{\zeta^2}\right) &= \left(5+4+3-\frac{60}{\zeta}\right)^2 \\ 100+\frac{7200}{\zeta^2} &= 144-\frac{1440}{\zeta}+\frac{3600}{\zeta^2} \\ 44\zeta^2-1440\zeta-3600 &= \\ \zeta &= \frac{180 \pm 30\sqrt{47}}{11} \end{aligned}$$

We take the positive solution. Computing $AP+BP+CP = 3\zeta - s = \frac{23+90\sqrt{47}}{11}$, which gives an answer of $23+90+47+11 = 171$.

15. $ABCD$ is a convex quadrilateral in which $\overline{AB} \parallel \overline{CD}$. Let U denote the intersection of the extensions of $\overline{AD}$ and $\overline{BC}$. Ω_1 is the circle tangent to line segment $\overline{BC}$ which also passes through A and D, and Ω_2 is the circle tangent to $\overline{AD}$ which passes through B and C. Call the points of tangency M and S. Let O and P be the points of intersection between Ω_1 and Ω_2. Finally, $\overline{MS}$ intersects $\overline{OP}$ at V. If $AB = 2$, $BC = 2005$, $CD = 4$, and $DA = 2004$, then the value of UV^2 is some integer n. Determine the remainder obtained when n is divided by 1000.

Answer: **039**. WLOG $M \in \overline{BC}$. Because $\triangle UAB \sim \triangle UDC$, $UA = 2004$ and $UB = 2005$. Now, by power of a point from U, $UM^2 = 2 \cdot 2004^2$ and $US^2 = 2 \cdot 2005^2$. Hence, $\frac{UM}{US} =$

$\frac{2004}{2005} = \frac{UA}{UB}$, implying that $\triangle UMS \sim \triangle UAB \sim \triangle UDC$. Also, $\frac{UA}{UM} = \frac{1}{\sqrt{2}} = \frac{UM}{UD}$ implying that $\triangle UAM \sim \triangle UMD$. Analogously, $\triangle UBS \sim \triangle USC$. Now $m\angle UMS = m\angle BAU = \pi - m\angle SAB$, hence, $ABMS$ is cyclic. Similarly, $SMCD$ is cyclic. Now, by the Radical Axis Theorem, $\overline{AM}, \overline{BS}$, and $\overline{OP}$ concur at T_1. Similarly, $\overline{CS}, \overline{DM}$, and $\overline{OP}$ concur at T_2. But $\angle USB \cong \angle SCU \cong \angle UDM$, so $\overline{ST_1} \parallel \overline{DM}$. Similarly, $\overline{MT_1} \parallel \overline{CS}$. T_1MT_2S is a parallelogram, hence V is the midpoint of $\overline{MS}$. Using similar triangles again, we find that $MS = 2\sqrt{2}$, from which $UV^2 = \frac{1}{4}\left(2UM^2 + 2US^2 - MS^2\right) = 2004^2 + 2005^2 - 2 \equiv 39 \pmod{1000}$.

1 Footnotes

Problems #1 and #13 were contributed by JGeneson. Problems #2 and #3 were inspired by problems from the 1999 AHSME and 2004 HMMT Guts #23 respectively. #10 is a more difficult version of a problem from AIME 1994. #12 was adapted from 2004 HMMT Guts #39. Finally, problem #15 is a short-answer case of a 1990 olympiad problem from China.