Mock AIME Solutions

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The following illustrate one possible approach to each problem. By no means are they the only possible solutions, nor are they necessarily the best solutions.

1. Answer: **680**. Consider independently the sums from each digit in ABC. Each digit $\{1, 2, \ldots, 9\}$ appears as A exactly $9 \cdot 8 = 72$ times since exactly 9 and 8 distinct digits are available to be B and C respectively. Each of these digits also appears as B and C $8 \cdot 8 = 64$ times since there are eight choices for A and C respectively. We may ignore the sum of the 0's, and we have $S = (1+2+\cdots+9)(100)(72)+(1+\cdots+9)(10)(64)+(1+\cdots+9)(1)(64) = 45 \cdot 7904 = 355,680$. Since we are told to divide S by 1000, the answer is 680.

2. Answer: **161**. Notice that the equation can be rewritten at $(x-15)^2+(y-20)^2 = 15^2+20^2-24^2 = 7^2$. It is clear that the possible (x, y) lie on a circle of radius 7 centered at $(15, 20)$. Consider $\frac{y}{x} = k$. This can be rewritten as $y = kx$. Thus, finding the maximum $\frac{y}{x}$ is equivalent to finding the line of maximum slope that passes through the origin and intersects the circle. This is the tangent to the circle that is nearer to the $+y$-axis. Let O denote the origin, P the center of the circle and T the point of tangency. By Pythagoras, $OP = 25$. OT has a length of $\sqrt{25^2 - 7^2} = 24$ since it is part of right triangle OTP. Let α be the angle formed by OP and the $+x$-axis and β the angle TOP. Then $\frac{m}{n}$ is

$$\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)} = \frac{\frac{4}{3}+\frac{7}{24}}{1-\frac{4}{3}\frac{7}{24}} = \frac{117}{44}$$

3. Answer: **110**. Without loss of generality, A, B, C, D, and E are points on the x-axis. We can assume further that P is in the plane, and that $A = (0, 0), B = (1, 0), C = (2, 0), D = (4, 0)$, and $E = (13, 0)$. Let $P = (x, y)$. Then

$$\begin{aligned} AP^2+BP^2+CP^2+DP^2+EP^2 &= x^2+(x-1)^2+(x-2)^2+(x-4)^2+(x-13)^2+5y^2 \\ &= 5x^2-40x+190+5y^2 \\ &= 5(x-4)^2+110+5y^2 \end{aligned}$$

since squares are non-negative, we choose $x = 4$ and $y = 0$ to give the sum its minimum of 110.

4. Answer: **801**. Note that $S = 7^0+7^1+\cdots+7^{2004} = (7^0+7^1+7^2+7^3)(7^0+7^4+\cdots+7^{2000})+7^{2004}$. But $1+7+49+343 = 400$, so that when we divide S by 1000 we care only about $\sum_{k=0}^{500} 7^{4k}$ modulo 10 and the extra term 7^{2004}. Since the sum contains $7^{4k} = (2401)^k \equiv 1 \pmod{10}$

for 501 values of k, $(7^0+7^1+7^2+7^3)(7^0+7^4+\cdots+7^{2000}) \equiv 400\cdot 501 \equiv 400 \pmod{1000}$. To determine $7^{2004} \pmod{1000}$, we note that $\phi(1000) = 400$ so that $7^{2004} \equiv 7^4 \equiv 401 \pmod{1000}$. Adding the two yields $S \equiv 801 \pmod{1000}$.

5. Answer: **118**. Let $a = \sqrt[3]{x}$ and $b = \sqrt[3]{20-x}$. We have $a+b = 2$ and $a^3+b^3 = 20 = (a+b)^3 - 3(a+b)(ab) = 8 - 6ab$. We find that $ab = -2 = a(2-a)$. We solve this for $a = 1 \pm \sqrt{3} = \sqrt[3]{x}$. Cubing both sides, we have $x = 10 \pm \sqrt{108}$. Hence, the answer is $10 + 108 = 118$.

   ALTERNATE SOLUTION

   Cube the given equation, and substitute the given recursively:

$$\begin{aligned} \left(\sqrt[3]{x}+\sqrt[3]{20-x}\right)^3 &= 20 + 3\left(\sqrt[3]{x}+\sqrt[3]{20-x}\right)\sqrt[3]{x(20-x)} = 8 \\ 20 + 6\sqrt[3]{x(20-x)} &= 8 \\ x(20-x) &= -8 \end{aligned}$$

   This is a quadratic and can easily be solved for $x = 10 \pm \sqrt{108}$, which gives the answer.

6. Answer: **504**. Let a_n be the number of ways the paperboy could deliver papers to n houses. We want to find a_{10}. We work out the small cases $a_1 = 2$, $a_2 = 4$, and $a_3 = 7$. Now consider the case $n \geq 4$. Either the paperboy delivers to the first house, after which there are a_{n-1} possible routes, or he skips the first house. If he skips the first house he may deliver to the second house, after which there are a_{n-2} routes, or he may skip the second house. If he skips the first and second houses, he must deliver to the third house, which leaves a_{n-3} possible routes. Hence, $a_n = a_{n-1} + a_{n-2} + a_{n-3}$. Now we have

$$\begin{aligned} a_4 &= a_3 + a_2 + a_1 = 7 + 4 + 2 = 13 \\ a_5 &= a_4 + a_3 + a_2 = 13 + 7 + 4 = 24 \\ a_6 &= a_5 + a_4 + a_3 = 24 + 13 + 7 = 44 \\ a_7 &= a_6 + a_5 + a_4 = 44 + 24 + 13 = 81 \\ a_8 &= a_7 + a_6 + a_5 = 81 + 44 + 24 = 149 \\ a_9 &= a_8 + a_7 + a_6 = 149 + 81 + 44 = 274 \\ a_{10} &= a_9 + a_8 + a_7 = 274 + 149 + 81 = 504 \end{aligned}$$

7. Answer: **320**. Suppose that k of the C's become A's and $4-k$ of the B's become A's. Then $6-k$ of the C's become B's and $1+k$ of the B's become C's. We have $k-1$ B's and $5-k$ C's to replace the 4 A's. This can be accomplished in

$$\begin{aligned} \sum_{k=0}^{4}\binom{6}{k}\binom{5}{4-k}\binom{4}{k-1} &= 1\cdot 5\cdot 0 + 6\cdot 10\cdot 1 + 15\cdot 10\cdot 4 + 20\cdot 5\cdot 6 + 15\cdot 1\cdot 4 \\ &= 0 + 60 + 600 + 600 + 60 = 1320 \end{aligned}$$

   ways. Dividing through by 1000 leaves a remainder of 320.

8. Answer: **177**. $\mathcal{P}$ and $\mathcal{P}'$ are similar; since the volume of the former is 8 times that of the latter, if follows that the plane passes through $\mathcal{P}$ halfway up the pyramid $\mathcal{P}$. Let Z be the apex of $\mathcal{P}$, and A', B', C', and D' the midpoints of $A'Z, B'Z, C'Z$, and $D'Z$ respectively. $A'B'C'D'$, the rectangular intersection of the plane and $\mathcal{P}$, has $A'B' = C'D' = 6$ and $B'C' = D'A' = 8$. Let O and O' denote the centers of $ABCD$ and $A'B'C'D'$ respectively. Since the height of $\mathcal{P}$ is 24, $OO' = 12$. By symmetry, the circumsphere of the frustum $\mathcal{F}$ is centered on OO'. Since for any point X on OO', we have $AX = BX = CX = DX$ and $A'X = B'X = C'X = D'X$, we need only find the point X such that $AX = A'X$. Suppose that $OX = x$ and $XO' = 12 - x$. By the Pythagorean theorem in 3-space, we have

$$\begin{aligned} AX = A'X \iff 6^2 + 8^2 + x^2 &= 3^2 + 4^2 + (12 - x)^2 \\ 100 + x^2 &= 25 + 144 - 24x + x^2 \\ x = \frac{69}{24} & \end{aligned}$$

Then $XT = 24 - \frac{69}{24} = \frac{507}{24} = \frac{169}{8}$, so the answer is $169 + 8 = 177$.

9. Answer: **576**. Consider the following algebra:

$$\begin{aligned} \frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{360}{pqr} &= 1 \\ pq + qr + rp + 360 &= pqr \\ 359 &= pqr - (pq + qr + rp) + ((p + q + r) - 26) - 1 \\ 385 &= (p - 1)(q - 1)(r - 1) \end{aligned}$$

Now consider the factorization $385 = 5 \cdot 7 \cdot 11$. Each term in the product $(p-1)(q-1)(r-1)$ has to divide 385. If any of $p-1$, $q-1$, $r-1$ contains two of the prime factors, then the sum $p+q+r$ cannot be 26 unless (WLOG) $p - 1 = 35, q - 1 = -11, r - 1 = -1$, but this is invalid since $r = 0$. Hence, $p - 1 = \pm 5, q - 1 = \pm 7, r - 1 = \pm 11 \implies p + q + r = 3 \pm 5 \pm 7 \pm 11 = 26$. By inspection, the only possibility is $p = 6, q = 8, r = 12$, which gives $pqr = 576$.

10. Answer: **113**. Overlay the complex number system with $P = 0 + 0i, A = 1 + 0i$, and $O = 1 + i$. The solutions to the equation $z^7 = i$ are seven points equally spaced around the unit circle centered at P. To translate these points to the heptagon $ABCDEFG$, we replace z with $(z - (1 + i))$, obtaining $(z - (1 + i))^7 = z^7 - \cdots + 8(-1 + i) = i$. The product we are interested is the magnitude of the product of the roots of this equation. Since this is a monic polynomial in z, the product of the solutions z_i is $8 - 7i$. Hence, we have $AP \cdot BP \cdot CP \cdot DP \cdot EP \cdot FP \cdot GP = |z_1 z_2 z_3 z_4 z_5 z_6 z_7| = |8 - 7i| = \sqrt{113}$. It follows that the answer is 113.

11. Answer: **006**. Consider the polynomial $f(x)$ defined by

$$f(x) = (x - 1)^{2004} = \sum_{n=0}^{2004} \binom{2004}{n} \cdot (-1)^n x^{2004-n}$$

Let $\omega^3 = 1$ with $\omega \neq 1$. We have

$$\frac{f(1) + f(\omega) + f(\omega^2)}{3} = \frac{(1-1)^{2004} + (\omega - 1)^{2004} + (\omega^2 - 1)^{2004}}{3}$$

$$
\begin{aligned}
&= \frac{1}{3}\sum_{n=0}^{2004}\binom{2004}{n}\cdot(-1)^n\cdot\left(1^{2004-n}+\omega^{2004-n}+(\omega^2)^{2004-n}\right)\\
&= \sum_{n=0}^{668}(-1)^n\binom{2004}{3n}
\end{aligned}
$$

where the last step follows in part from the fact that the only integers n for which $1^n+\omega^n+\omega^{2n}$ is non-zero are multiples of three, where the expression is always equal to 3. WLOG, $\omega-1=\sqrt{3}\cdot\frac{-\sqrt{3}+i}{2}$ and $\omega^2-1=\sqrt{3}\cdot\frac{-\sqrt{3}-i}{2}$. Both expressions, when raised to the 2004-th power, become 3^{1002}, as their complex factors are two of the 12-th roots of unity and $2004=12\cdot 167$. Hence,

$$
S=\sum_{n=0}^{668}(-1)^n\binom{2004}{3n}=2\cdot 3^{1001}
$$

In finding $2\cdot 3^{1001} \pmod{1000}$, we note that $3^{\phi(500)}\equiv 3^{200}\equiv 1 \pmod{500}$ so that $3^{1001}\equiv 3 \pmod{500}$. Hence, we may write $2\cdot 3^{1001}=2\cdot(3+500k)\equiv 6 \pmod{1000}$ for some integer k. It follows that the answer is 6.

12. Answer: **338**. By the reflection, we have $B'E=BE=23$. Because $ABCD$ is a rectangle, we have $m\angle C'AE=m\angle C'B'E=\frac{\pi}{2}\implies C'AB'E$ is cyclic with diameter $C'E\implies \angle B'C'A\cong\angle B'EA\cong\angle AB'C'\implies\triangle AB'C'$ is isosceles with $AB'=AC'=5$. It would suffice to determine $C'E$ as this would eventually yield both sides of $ABCD$.

    Let ω denote the circumcircle of $AB'EC'$. Consider the point P on the minor arc $B'E$ of ω such that $AP=23$ and $PE=5$. $APEC'$ is an isosceles trapezoid with $m\angle C'AE=m\angle C'PE=\frac{\pi}{2}$. Let $C'E=x$. Then by Pythagoras, $C'B'=AE=\sqrt{x^2-25}$, but by Ptolemy's Theorem applied to this trapezoid,

$$
23x+25=x^2-25
$$

    from which we find $x=25$ or -2. Taking $C'E=x=25$, we obtain $AE=\sqrt{625-25}=10\sqrt{6}$ and $C'B'=\sqrt{25^2-23^2}=4\sqrt{6}$.

    Now we have $AB=AE+EB=10\sqrt{6}+23$ and $C'B'=BC=4\sqrt{6}$ so that the area of $ABCD$ is $240+92\sqrt{6}$, which yields an answer of $240+92+6=338$.

13. Answer: **443**. We have $R_0=10, R_1=-2$ and $R_2=\frac{1}{7}\cdot(64-2(-2)+9(10))=\frac{158}{7}$. We solve for 64 in terms of the sequence, obtaining

$$
64=7R_{n+3}+2R_{n+2}-9R_{n+1}=7R_{n+2}+2R_{n+1}-9R_n
$$

    for all integers $n\geq 0$. The characteristic equation of $\{R_n\}_{n\geq 0}$ is now seen to be $7x^3+2x^2-9x=7x^2+2x-9$ or $7x^3-5x^2-11x+9=0$. The rational root test can be applied to facilitate guess and check, which produces the factorization $(x-1)^2(7x+9)=0$. This implies that we have $R_n=a\cdot n+b+c\cdot\left(\frac{-9}{7}\right)^n$. We solve for a,b,c by checking the first three terms:

$$
\begin{aligned}
R_0 &= b+c=10\\
R_1 &= a+b-\frac{9c}{7}=-2\\
R_2 &= 2a+b+\frac{81c}{49}=\frac{158}{7}
\end{aligned}
$$

This can be accomplished by algebra, but we should always check for simple solutions. Intuitively, it seems that c should be a multiple of 7. Plugging in $c = 7$ makes it easy to find the unique solution $(a, b, c) = (4, 3, 7)$. Hence, we are asked to compute the sum

$$\begin{aligned} S = \sum_{i=0}^{\infty} \frac{R_i}{2^i} &= \sum_{n=0}^{\infty} \frac{4n + 3 + 7 \cdot \left(\frac{-9}{7}\right)^n}{2^n} \\ &= 4\sum_{n=0}^{\infty} \frac{n}{2^n} + 3\sum_{n=0}^{\infty} \frac{1}{2^n} + 7\sum_{n=0}^{\infty} \left(\frac{-9}{14}\right)^n \end{aligned}$$

By the formula for the sum of an infinite geometric sequence, we have $\sum_{n=0}^{\infty} \frac{1}{2^n} = 2$ and $\sum_{n=0}^{\infty} \left(\frac{-9}{14}\right)^n = \frac{14}{23}$. Let $T = \sum_{n=0}^{\infty} \frac{n}{2^n}$. We telescope T with itself, finding

$$\begin{aligned} T &= 2T - T \\ &= \left(\frac{1}{1} + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \cdots\right) - \left(\frac{0}{1} + \frac{1}{2} + \frac{2}{3} + \frac{3}{8} + \cdots\right) \\ &= \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \\ &= 2 \end{aligned}$$

so that $S = 8 + 6 + \frac{98}{23} = \frac{420}{23}$, which gives an answer of $420 + 23 = 443$.

ALTERNATE SOLUTION (Due to Yoni Levy)

Divide the given by 2^n, obtaining $7\frac{R_n}{2^n} = \frac{64}{2^n} - \frac{R_{n-1}}{2^{n-1}} + \frac{9}{4}\frac{R_{n-2}}{2^{n-2}}$. Let us sum this equation from 2 to ∞. That is,

$$\begin{aligned} \sum_{n=2}^{\infty} 7\frac{R_n}{2^n} &= \frac{64}{2^n} - \frac{R_{n-1}}{2^{n-1}} + \frac{9}{4}\frac{R_{n-2}}{2^{n-2}} \\ 7\sum_{n=2}^{\infty} \frac{R_n}{2^n} &= 64\sum_{n=2}^{\infty} - \sum_{n=1}^{\infty} \frac{R_n}{2^n} + \frac{9}{4}\sum_{n=0}^{n} \frac{R_n}{2^n} \\ 7(S-9) &= 32 - (S - 10) + \frac{9}{4}S \end{aligned}$$

This equation can readily be solved for $S = \frac{420}{23}$, which leads to the correct answer.

ALTERNATE SOLUTION (Due to Daniel J. Hermes / Pork_Chop8)

Note that we may write

$$\begin{aligned} 7R_{n+2} + 9R_{n+1} &= 64 + 7R_{n+1} + 9R_n \\ &= 64 + (64 + 7R_n + 9R_{n-1}) \\ &\cdots \\ &= 64 \cdot (n+1) + 7R_1 + 9R_0 = 64n + 140 \end{aligned}$$

Therefore,

$$
\begin{aligned}
23S = 23\sum_{i=0}^{\infty}\frac{R_i}{2^i} &= 2\cdot 7\sum_{i=0}^{\infty}\frac{R_i}{2^i} + 9\sum_{i=0}^{\infty}\frac{R_i}{2^i} \\
&= 2\cdot 7\left(\left(\sum_{i=2}^{\infty}\frac{R_i}{2^i}\right) + \frac{R_1}{2^1} + \frac{R_0}{2^0}\right) + 9\left(\left(\sum_{i=1}^{\infty}\frac{R_i}{2^i}\right) + \frac{R_0}{2^0}\right) \\
&= 2\cdot 7\sum_{i=0}^{\infty}\frac{R_{i+2}}{2^{i+2}} + 126 + 9\sum_{i=0}^{\infty}\frac{R_{i+1}}{2^{i+1}} + 90 \\
&= 7\sum_{i=0}^{\infty}\frac{R_{i+2}}{2^{i+1}} + 9\sum_{i=0}^{\infty}\frac{R_{i+1}}{2^{i+1}} + 216 = \sum_{i=0}^{\infty}\frac{7R_{i+2}+9R_{i+1}}{2^{i+1}} + 216 \\
&= \sum_{i=0}^{\infty}\frac{64i+140}{2^{i+1}} + 216 \\
&= 32\sum_{i=0}^{\infty}\frac{i}{2^i} + 70\sum_{i=0}^{\infty}\frac{1}{2^i} + 216 = 32\cdot 2 + 70\cdot 2 + 216 = 420
\end{aligned}
$$

From which it follows that $S = \frac{420}{23}$ and the answer is $420 + 23 = 443$.

14. Answer: **352**. Suppose that the two types of keys are A and B. Let the 12 character string $X = X_1X_2X_3\dots X_{12}$ represent a generic keychain. Define $R_i(X_1\dots X_{12}) = X_{i+1}\dots X_{12}X_1\dots X_i$ for $i \in \mathbb{Z}_{12}$ to represent a rotation of i keys. We will argue that the answer should be given by

$$\frac{\sum_{i=0}^{11} c_i}{12}$$

where c_i is the number of strings that remain fixed under R_i. Suppose the string X repeats every n (where n is minimal) characters. Obviously $n|12$ or else the string repeats every $\gcd(n,12) < n$ characters. We want to show that

$$\sum_{i=0}^{11} c_i$$

counts X exactly $\frac{12}{n}$ times, since there are exactly this many rotations R_i that fix X. This is the case, however, since R_{kn} fixes X for integers k, and exactly $\frac{12}{n}$ such k exist for which $kn \in \mathbb{Z}_{12}$. Based on our definition of c_i, we know that the string X is counted in each c_{kn} and only these c_i. But c_i is the number of strings that are fixed under R_i, which means that these strings must repeat every i, which implies that they *must* repeat every $\gcd(i,12)$. Since each such string is a block of $\gcd(i,12)$ characters copied $\frac{12}{\gcd(i,12)}$ times, there are exactly $c_i = 2^{\gcd(i,12)}$ such strings.

Thus the summation $\sum_{i=0}^{11} c_i$ counts every string 12 times and there are

$$\frac{2^{12}+2^1+2^2+2^3+2^4+2^1+2^6+2^1+2^4+2^3+2^2+2^1}{12} = \frac{4224}{12} = 352$$

different keychains. This is the basic idea behind the Pólya-Redfield method of counting distinct, rotationally-independent strings.

ALTERNATE SOLUTION

Let the types of keys be A and B. Consider the keychain X, represented by a string of 12 A's and B's. We will count the number of distinct keychains with $0, 1, 2, \ldots, 11$, and 12 type B keys. Since there exists a bijection (A's to B's and B's to A's) between the case with n B's and $12 - n$ B's, we need only consider 0, 1, 2, 3, 4, 5, and 6 type B's.

Let S_n denote the number of 12-character strings with n B's and $12 - n$ A's. Let $A_{n,m} \subseteq S_n$ denote the subset of S_n that contains strings fixed under rotation[1] by m (not necessarily minimal) characters. We will call a rotation that leaves a string unchanged a *fixing rotation*.

Obviously, there is one string with 0 B's, and there is one string with a single B that has 12 rotational positions. The case with 2 B's is a question of how far apart the B's are, which has 6 possibilities. The case with 5 B's has a rotation iff all of the string is all B's, which is a contradiction as there are only 5 B's. Hence, any string with 5 B's has 12 non-identical rotational positions, and it follows that there are $\frac{C(12,5)}{12} = 66$ such rotationally independent strings.

In the case with 3 B's, the possible fixing rotations are the indentity rotation and a rotation of 4 characters. There are $|S_3| - |A_{3,4}| = C(12, 3) - C(4, 1) = 220 - 4 = 216$ strings fixed only under the identity rotation and $|A_{3,4}| = C(4, 1) = 4$ strings with 3 B's fixed only under a rotation of 4 characters. Hence, there are $\frac{216}{12} + \frac{4}{4} = 18 + 1 = 19$ rotationally independent strings with 3 B's.

In the case with 4 B's, the possible fixing rotations are the indentity rotation and rotations by 3 and 6 characters. There are $|S_4| - |A_{4,6}| = C(12, 4) - C(6, 2) = 495 - 15 = 480$ strings fixed only under the identity rotation, $|A_{4,6}| - |A_{4,3}| = C(6, 2) - C(3, 1) = 15 - 3 = 12$ strings fixed only under a rotation by 6 characters, and $|A_{4,3}| = C(3, 1) = 3$ strings fixed only under rotation by 3 characters. Hence, there are $\frac{480}{12} + \frac{12}{6} + \frac{3}{3} = 40 + 2 + 1 = 43$ rotationally independent strings with 4 B's.

For the case with 6 B's, the possible fixing rotations are the identity rotation and rotations of 2, 4, and 6. A rotation of 3 characters cannot hold a string fixed since this would require that there were a multiple of 4 B's, a contradiction. The number of strings that are fixed only under the identity rotation is given by $|S_6| - |A_{6,6}| - |A_{6,4}| + |A_{6,2}| = C(12, 6) - C(6, 3) - C(4, 2) + C(2, 1) = 924 - 20 - 6 + 2 = 900$. The number of strings fixed under 6-character rotation is given by $|A_{6,6}| - |A_{6,2}| = C(6, 3) - C(2, 1) = 20 - 2 = 18$, and the numbers of strings fixed under rotations of 4 and 2 characters are given by $C(4, 2) - C(2, 1) = 6 - 2 = 4$ and $C(2, 1) = 2$ respectively. Hence, $|S_6| = \frac{900}{12} + \frac{18}{6} + \frac{4}{4} + \frac{2}{2} = 75 + 3 + 1 + 1 = 80$.

Therefore, there are $2 \cdot (1 + 1 + 6 + 19 + 43 + 66) + 80 = 352$ distinct possible keychains.

15. Answer: **141**. It follows from $2 \cos B = \cos A + \cos C$ that $\cos A, \cos B, \cos C$ is an arithmetic progression. It also follows that

$$3 \cos B = \cos A + \cos B + \cos C = 1 + \frac{r}{R} = \frac{21}{16}$$

[1] By "rotation" we mean taking the string $X_1X_2 \ldots X_{12}$ to $X_{m+1} \ldots X_{12}X_1 \ldots X_m$.

so we may set $\cos A = \frac{7}{16} + k, \cos B = \frac{7}{16}, \cos C = \frac{7}{16} - k$. We substitute these into another famous trig identity,

$$\begin{aligned} \cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C &= 1 \\ 3 \cdot \left(\frac{7}{16}\right)^2 + 2k^2 + 2 \cdot \frac{7}{16}\left(\left(\frac{7}{16}\right)^2 - k^2\right) &= 1 \\ \frac{18}{16} \cdot 16^3 \cdot k^2 + 7^2(48+14) &= 16^3 \\ 2 \cdot 3^2 \cdot 16^2 \cdot k^2 = 1058 &= 2 \cdot 23^2 \\ k &= \pm\frac{23}{48} \end{aligned}$$

So we have $\cos A = \frac{11}{12}, \cos B = \frac{7}{16}$, and $\cos C = \frac{-1}{24}$, which imply $\sin A = \frac{\sqrt{23}}{12}, \sin B = \frac{3}{16}\sqrt{23}$, and $\sin C = \frac{5}{24}\sqrt{23}$ respectively. Finally,

$$[ABC] = 2R^2 \sin A \sin B \sin C = 2 \cdot 16^2 \cdot \left(\frac{\sqrt{23}}{12}\right)\left(\frac{3}{16}\sqrt{23}\right)\left(\frac{5}{24}\sqrt{23}\right) = \frac{115\sqrt{23}}{3}$$

which gives an answer of $115 + 23 + 3 = 141$.

1 Footnotes

Problems #6 and #7 were inspired by AIME 2001/14 and a recent AMC-12 problem respectively. The first solution to #14 is essentially the Pólya-Redfield method; I encountered this in a later chapter in Daniel A. Marcus: *Combinatorics*. It was pure coincidence that #11 was a similar problem to #17 on the Mock AMC A - I had written this problem before seeing the latter.