TJMC #1

NO CALCULATORS, 40 Minutes

1.1 - Compute A + B + C given the following equations:

$$\begin{aligned} 2A + B + C &= 8 \\ A + 2B + C &= 6 + \sqrt{3} \\ A + B + 2C &= 10 - \sqrt{3} \end{aligned}$$

1.2 - Compute the positive integer remainder when $1^{2003} + 2^{2003} + \cdots + 2002^{2003} + 2003^{2003}$ is divided by 10.

1.3 - $f(x)$ is a cubic function in x such that its three roots are 1, 2002, and 2003. Compute the value of $\frac{f(2000)}{f(2001)}$.

1.4 - ABCDEF is a regular hexagon of side length $\frac{2002}{2003}$. Compute the value of $(\frac{AC}{AD})^2$.

1.5 - Simplify the following expression as far as possible, while keeping your answer exact:

$$\sqrt[3]{\sqrt{18 + \sqrt{128}} + \sqrt{18 - \sqrt{128}}} + \sqrt[3]{\sqrt{18 + \sqrt{128}} - \sqrt{18 - \sqrt{128}}}$$

1.6 - $f(x)$ is a monic, cubic function with three distinct roots. Let the a, b, and c be these roots. Compute $f(1)$ given the three following equations relating a, b, and c.

$$\begin{aligned} a + b + c &= 1 \\ ab + ac + bc &= 1002 \\ abc &= 2003 \end{aligned}$$

1.7 - Find all order pairs (a, b) for real a and b such that

$$a + b - 10\sqrt{a + b} + 25 = 9$$
$$a - b - 10\sqrt{a - b} + 49 = 25$$

1.8 - ABCD is a convex quadrilateral inscribed in circle O such that AB = 5, BC = 8, and CD = 7. Chord AD is extended beyond A to point P such that AP = 2. A tangent to circle O from P is of length 4. Compute the numerical value of BD^2.