TJMC #2 SOLUTIONS

NO CALCULATORS, 40 Minutes

2.1 - John has a biscuit. Joe asks John if he can have half of his biscuit. John says ok, and gives him "half" of his biscuit. Joe complains, saying "Your half is bigger than mine." John agrees, and tearing off a third of his "half" of the biscuit and eating it. John then shows that the two new "halves" are now equal. What fraction of the biscuit did John give to Joe?

ANSWER: $\frac{2}{5}$. Let the entire biscuit be 1, and let John give Joe x amount of biscuit. From the story, we find that $\frac{2}{3}(1-x) = x$. Solving for x we obtain $x = \frac{2}{5}$.

2.2 - How many zeros are at the end of 2003!?

ANSWER: 499. This problem boils down to computing the N such that $2003! = k * 10^N$, where k is not a multiple of 10. Since $10 = 2 * 5$, we can examine the prime factorization of 2003!. There are more 2's than 5's, so we need to find the number of 5's in the prime factorization. For 2003!, this is $\sum_{n=1}^{\infty} int(\frac{2003}{5^n}) = 400 + 80 + 16 + 3 + 0 + 0 + \cdots = 499$.

2.3 - How many integers between 1 and 1,000,000,000,000 inclusive are perfect squares, perfect cubes, or an integer raised to the fourth power?

ANSWER: 1,009,900. Of course all integers that are another integer to the fourth power are also squares, so we can ignore the stipulation that we are looking for four powers. The number of integers that are squares or cubes is the number of squares + the number of cubes - their intersection, which is all of the sixth powers. There are $\sqrt{1,000,000,000,000} = 1,000,000$ perfect squares, $\sqrt[3]{1,000,000,000,000} = 10,000$ perfect cubes, and $\sqrt[6]{1,000,000,000,000} = 100$ sixth power's. Thus, there are $1,000,000 + 10,000 - 100 = 1,009,900$ numbers that are squares or cubes.

2.4 - Let $f(x)$ be a quadratic function passing through points $(1, 10)$, $(2, 0)$ and $(3, -5)$. If $g(x) = a(x-1)(x-2) + b(x-1)(x-3) + c(x-2)(x-3)$, and $g(x) = f(x)$ for all x, compute the numerical value of $a + b + c$.

ANSWER: $\frac{5}{2}$. Substituting x = 1, 2, and 3 produces the following equations and values for a, b, and c:

$$\begin{aligned} 10 = f(1) &= a(1-1)(1-2) + b(1-1)(1-3) + c(1-2)(1-3) = 2c \Longrightarrow c = 5 \\ 0 = f(2) &= a(2-1)(2-2) + b(2-1)(2-3) + (2-2)(2-3) = -b \Longrightarrow b = 0 \\ -5 = f(3) &= a(3-1)(3-2) + b(3-1)(3-3) + c(3-2)(3-3) = 2a \Longrightarrow 2a = \tfrac{-5}{2} \end{aligned}$$

Which gives $a + b + c = \frac{5}{2}$.

2.5 - Points A through H are selected such that:

i) Angles ABC, CAD, DAE, AEF, AHF, and HFG are all right angles.
ii) AB = 3, BC = 4, CD = 13, DE = 20, FE = 20, AH = 31, and GH = 15.

Compute the numerical value of GF^2.

ANSWER: 30. After a well drawn diagram, we can see that $AC = 5 \Rightarrow AD = 12 \Rightarrow AE = 16 \Rightarrow AF = 34 \Rightarrow FH = \sqrt{195} \Rightarrow GF = \sqrt{30} \Rightarrow GF^2 = 30$.

2.6 - A polynomial with integer coefficients passes through the points $(-2, 1)$ and $(2003, k)$. What is the largest possible k such that $k \leq 100,000$?

ANSWER: 98,246. Because of the property $\frac{a^n - b^n}{a-b}$, we see that $k - 1$ must be a multiple of $a - b$. In this case, $a = 2003$, and $b = -2$. We can now say that $k = 1 + 2005n$, for some integer n. Through long division, we find that $100,000 = 2005 * 49 + 1755$. Since $2005 * 49 + 1 \leq 100,000$, we conclude that $k = 49 * 2005 + 1 = 98,246$.

2.7 - Determine all real x such that:

$$x = \frac{3}{\frac{3}{x} + 3 + \frac{x}{3}} + 3 + \frac{\frac{3}{x} + 3 + \frac{x}{3}}{3}$$

ANSWER: $\frac{9\pm\sqrt{153}}{4}$, $\frac{-9\pm6\sqrt{2}}{2}$. We notice that the expression can be generated through recursion in the expression $x = \frac{3}{x} + 3 + \frac{x}{3}$ which is the quadratic $2x^2 - 9x - 9 = 0$. We solve for x in this simpler expression and find $x = \frac{9\pm\sqrt{153}}{4}$. Working in the original, we remove all of the fractions and obtain a quartic. Using our recursive solutions, we can factor the quartic easily, and find the other two values of x.

$$x = \frac{9x}{x^2+9x+9} + 3 + \frac{x^2+9x+9}{9x}$$

$$x(9x)(x^2+9x+9) = 81x^2 + 3(9x)(x^2+9x+9) + (x^2+9x+9)^2$$

$$8x^4 + 36x^3 - 342x^2 - 405x - 81 = 0$$

$$(2x^2 - 9x - 9)(4x^2 + 36x + 9) = 0$$

$$4x^2 + 36x + 9 = 0 \Longrightarrow x = \frac{-9\pm6\sqrt{2}}{2}$$

2.8 - ABCD is a cyclic quadrilateral inscribed in circle O. AB is extended beyond B, and DC is extended beyond C. AB and DC intersect at P. A point M is selected on O such that MD = 4 and MC = 9. Another point N is selected such that NC = 9. Given that AB = 4, and that PC + 1 = PB = DC - 1, find the sum of the two possible lengths of ND.

ANSWER: $\frac{279}{32}$. Let $PB = x$. Power of a point from P yields $x(x+4) = (x-1)(2x)$, from which we find $x = 6$ or 0. Since $PC + 1 = PB = x$, we find that $x = 6$, otherwise PC would be a negative length. Thusly, we find that $CD = 7$. Since both MC and NC are greater in length than CD, we can say that $mDMC = mDNC = \gamma$. Law of Cosines from M yields:

$$7^2 = 4^2 + 8^2 - 2*4*8*cos(\gamma)$$
$$cos(\gamma) = \tfrac{31}{64}$$

A second application of the Law of Cosines, this time from N yields:

$$49 = 81 + ND^2 - \tfrac{279}{32}ND$$
$$ND^2 - \tfrac{279}{32}ND + 32 = 0$$

Without solving for both of the values of ND that satisfy the quadratic, we can see that their sum is $\frac{279}{32}$, and we are done.