TJMC #3 SOLUTIONS

NO CALCULATORS, 40 Minutes

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3.1 - In Puzzletopia, there are two kinds of people, those who always lie, known as Fibsters, and those who always tell the truth, known as Beholders. At a big meeting of 2003 of Puzzletopians, one of them says, "We are all Fibsters." A second says, "One of us is a Beholder." A third Puzzletopian says, "No, two of us are Beholders." This goes on until the last one says "All but one of us are Beholders." Given that at least one of them is telling the truth, how many Fibsters are there?

ANSWER: 2002. The number of Fibsters can be any integer from 0 to 2003. All of the claim that there are 1 to 2003 Fibsters. Clearly, at most, one of them is correct, which means that there is at most one Beholder. Since we are told that there is at least one Beholder, we know that there is one Beholder, which implies that there are 2002 Fibsters.

3.2 - A regular octagon has a perimeter of 80. What is its area?

ANSWER: $200 + 200\sqrt{2}$. There are four corner triangles that have a hypotenuse of 10 and two angles of 45, there is a center square of side length 10, and there are 4 rectangles that have a length of 10 and a width of $5\sqrt{2}$. The four triangles' areas sum to 100, the square has an area of 100, and the four rectangles' areas sum to $200\sqrt{2}$.

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3.3 - What is the lowest degree that a polynomial passing through the points $(0, 4)$, $(1, 1)$, $(2, 0)$, $(3, 1)$, and $(4, 4)$ can have?

ANSWER: 2. Shifting each of the points left 2 units yields points that appear to lie on $y = x^2$. Checking shows this to be accurate. Since all five of the points to not lie on a single line, we conclude that the lowest possible degree is 2.

3.4 - How many positive integers satisfy both of the following:

i) All of the digits are either 1, 2, 3, or 4.
ii) The sum of the integer's digits is 8.

ANSWER: 108. Through recursion, where $f(n)$ is the number of integers whose digits are 1, 2, 3, or 4 and sum to n, we have $f(n) = f(n-1) + f(n-2) + f(n-3) + f(n-4)$, where $n \geq 5$. We also know through scribbling a few numbers down that $f(1) = 1, \quad f(2) = 2, \quad f(3) = 4, \quad f(4) = 8$. Adding, we find that:

$$f(5) = 1 + 2 + 4 + 8 = 15$$
$$f(6) = 2 + 4 + 8 + 15 = 29$$
$$f(7) = 4 + 8 + 15 + 29 = 56$$
$$f(8) = 8 + 15 + 29 + 56 = 108$$

3.5 - $ABCD$ is a tetrahedron such that $AB = 11, AC = 12, AD = 13, BC = 14, CD = 15$, and $DB = 16$. Let M be the midpoint of AB and N be the midpoint of CD. Compute the numerical value of MN^2.

ANSWER: $\frac{419}{4}$. We will repeatedly use the property that for a triangle of side lengths a, b, and c, where M is the median of side c, $CM^2 = \frac{2a^2+2b^2-c^2}{4}$. We have the following:

$$MC^2 = \frac{2*14^2+2*12^2-11^2}{4} = \frac{559}{4}$$
$$MD^2 = \frac{2*16^2+2*13^2-11^2}{4} = \frac{729}{4}$$
$$MN^2 = \frac{2MC^2+2MD^2-15^2}{4} = \frac{2\frac{559}{4}+2\frac{729}{4}-15^2}{4} = \frac{419}{4}$$

3.6 - The sum

$$\sum_{k=1}^{2003} kcis\left(\frac{360k^o}{2003}\right)$$

Can be expressed in the form $A + Bi$. Compute A. Note $cis\alpha = cos\alpha + isin\alpha$, where $i = \sqrt{-1}$.

ANSWER: $\frac{2003}{2}$ or 1001.5. Expanding the summation produces $0cis(0^o) + 1cis(\frac{360^o}{2003}) + 2cis(\frac{720^o}{2003}) + \cdots + 2002cis(\frac{2002*360^o}{2003}) + 2003cis(360^o)$. We can rewrite this as:

$$\sum_{k=0}^{2002}\sum_{i=0}^{k} cis\left(\frac{360*(2003-i)^o}{2003}\right)$$

This is more useful because that can be pictured as 2003 vectors to equally spaced points on a circle in the complex plane. The sum will then be 2003 times the "average" vector, which is the center. We cannot easily find the exact center, but we can determine the real portion of it. Note that two of the "points" that we have are $(0, 0)$ and $(1, 0)$. The perpendicular bisector of any chord of a circle passes through the center, and the chord formed by these two points is no exception. The perpendicular bisector is $x = \frac{1}{2}$, and we obtain $a = \frac{2003}{2}$.

3.7 - Find all positive integral x such that $x^4 - x^2 + 9012345$ is a multiple of 73.

ANSWER: 5, 7, 66, 68, 78, and 80. First, we divide 9012345 by 73 to reduce it to a manageable size, namely 57. We have $x^4 - x^2 + 57 \equiv 0(mod\ 73)$. Since we work with integral x, we can replace $-x^2$ with $-74x^2$. We now complete the square by adding $37^2 - 57$ to both sides. We now have $(x^2 - 37)^2 \equiv 1312(mod\ 73)$. 1312 fairly large, so we reduce it to 71 by long division. 71 is not a perfect square, but we see that 144 is. This is a difference of 73, so replacing 71 with 144 is valid. Now we can take the square root and obtain $x^2 - 37 \equiv \pm 12(mod\ 73)$. Adding 37 to both sides yields $x^2 \equiv 25$ or $49(mod\ 73)$. Taking the square root again yields $x \equiv \pm 5$ or $\pm 7(mod\ 73)$. We seek positive integral x so we add 73 to all four of these x's to find the other satisfactory x.

3.8 - Three circles ω_1, ω_2, and ω_3 are externally tangent and have radii of $\frac{1}{5}, \frac{1}{7}$, and $\frac{1}{11}$ respectively. Two other circles Ω_1, and Ω_2 are each tangent to all three ω_n. Specifically, the radius of Ω_2 is greater than the radius of Ω_1. Compute the sum of the radii of Ω_1 and Ω_2.

ANSWER: $\frac{4\sqrt{167}}{139}$. Both Ω_1 and Ω_2 can be found as cases of the Descartes Circle Theorem. Let $\frac{1}{x}$ be the radius of Ω_1. Then we have:

$$2(5^2 + 7^2 + 11^2 + x^2) = (x + 5 + 7 + 11)^2$$
$$2x^2 + 390 = x^2 + 46x + 529$$

$$x = 23 \pm 2\sqrt{167} \Longrightarrow x = 23 + 2\sqrt{167}$$
$$\tfrac{1}{x} = \tfrac{2\sqrt{167}-23}{139}$$

Letting $\frac{1}{y}$ be the radius of Ω_2, we have:

$$2(5^2 + 7^2 + 11^2 + y^2) = (-y + 5 + 7 + 11)^2$$
$$2y^2 + 390 = y^2 - 46y + 529$$
$$y = -23 \pm 2\sqrt{167}$$
$$\tfrac{1}{y} = \tfrac{2\sqrt{167}+23}{139}$$

Thusly, adding the two radii produces the answer.