TJMC #4 SOLUTIONS

NO CALCULATORS, 40 Minutes

4.1 - Triangle ABC has an area of 2003. Let D, E, and F be the midpoints of BC, AC, and AB respectively. Compute the area of ΔDEF.

ANSWER: $\frac{2003}{4}$. Through the use of similar triangles, one can deduce that ΔDEF is the same as ΔABC, except that its sides are half of the lengths of the corresponding sides on ΔABC. This gives $\frac{1}{4}$ of the area.

4.2 - There is one two digit positive integer that is 3 more than a multiple of 2, 4 more than a multiple of 3, 5 more than a multiple of 4, and 6 more than a multiple of 5. Find this number.

ANSWER: 61. The givens also imply that the two digit integer is 1 more than a multiple of 2, 1 more than a multiple of 3, 1 more than a multiple of 4, and 1 more than a multiple of 5. The least common multiple of 2, 3, 4, and 5 is 60, so the two digit integer is one more than this.

4.3 - Find the number of real ordered pairs (x, y) such that both of the following are satisfied:

$$i)\ x^2 + 4y = 4$$
$$ii)\ y^2 + 2x = 16$$

ANSWER: 2. Adding the two equations yields $x^2 + 2x + y^2 + 4x = 20$, which can be rexpressed as $(x + 1)^2 + (y + 2)^2 = 25$, which is clearly a circle of radius 5 centered at $(-1, -2)$. Solving for y in $i)$ yields $y = 1 - \frac{x^2}{4}$, which is clearly a parabola with a vertex at $(0, 1)$, which is inside the circle. It is then easy to see that the parabola and the circle will intersect exactly twice.

4.4 - $f(x) = x^3 - 21x^2 + 15x - 8$ is a polynomial with three roots ω_1, ω_2, and ω_3. Compute the numerical value of $(\omega_1)^3 + (\omega_2)^3 + (\omega_3)^3$.

ANSWER: 8,340. The coefficients represent various expressions involving the three roots. In particular:

$$i) \quad 21 = \omega_1 + \omega_2 + \omega_3$$
$$ii) \quad 15 = \omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3$$
$$iii) \quad 8 = \omega_1\omega_2\omega_3$$

Squaring $i)$ and subtracting twice $ii)$ yields:

$$iv) \quad 411 = (\omega_1)^2 + (\omega_2)^2 + (\omega_3)^2$$

Multiplying $i)$ and $iv)$ gives:

$$v) \quad 8631 = (\omega_1)^3 + (\omega_2)^3 + (\omega_3)^3 + [(\omega_1)^2\omega_2 + (\omega_1)^2\omega_3 + (\omega_2)^2\omega_1 + (\omega_2)^2\omega_3 + (\omega_3)^2\omega_1 + (\omega_3)^2\omega_2].$$

From which we need only to determine and subtract the expression within the brackets. It turns out that the unwanted portion is $i)$ times $ii)$ minus three times $iii)$, or $21 * 15 - 3 * 8 = 291$. We can then determine the desired expression to be 8340.

4.5 - $a_1, a_2, \ldots$ are real numbers such that $n^2(a_1 + a_2 + \cdots + a_n) = 10 + 30(a_{n+1} + a_{n+2} + \cdots + a_{30})$ for positive, integral n. Compute a_{14}.

ANSWER: $\frac{-279}{44974}$. Define $S_n = a_1 + a_2 + \cdots + a_n$, allowing us to rexpress the given as $n^2 S_n = 10 + 30(S_{30} - S_n)$, in which we can solve for S_n. Arriving at $S_n = \frac{10+30S_{30}}{n^2+30}$, we let $n = 30$, and solve for S_{30}, which is $\frac{1}{90}$. We now have $S_n = \frac{31}{3(n^2+30)}$. Since $a_{14} = S_{14} - S_{13}$, we can compute: $a_{14} = \frac{31}{3(226)} - \frac{31}{3(199)} = \frac{-279}{44974}$.

4.6 - Thomas is walking in the coordinate plane. Starting at the origin, he walks to $(6, 0)$. He then turns left α degrees and then walks a fraction x of the distance he just walked to get to $(9, 4)$. He turns left α degrees again and walks x of the distance he walked to get to another point. This process of turning and walking a smaller amount continues on and on. Thomas eventually begins to converge on a point. Find this point.

ANSWER: $(\frac{108}{25}, \frac{144}{25})$. We move the coordinate plane into the complex plane such that the x and real axes align, and the origins are on top of each other. We represent each move as a vector ν_n stemming from the origin. We have $\nu_1 = 6 + 0i$. We see that the next vector is the first one after a rotation and then scaling. This is just like multiplying in the complex plane. Specifically, $\nu_2 = 3 + 4i$, but we prefer to represent this as $\nu_2 = \nu_1 * \tau$, where $\tau = \frac{3+4i}{6}$. We prefer this form because although we do not know what ν_3 is componentwise, we do know that $\nu_3 = \nu_2 * \tau = \nu_1 * \tau^2$. Ultimately, $\nu_n = \nu_1 * \tau^{n-1}$. We wish to find the sum $\nu_1 + \nu_2 + \cdots$. This is clearly a geometric series, in which the magnitude of τ is less than 1. Consequently, the infinite sum is $\frac{6}{1-\tau} = \frac{6}{\frac{1}{2} - \frac{2i}{3}}$. Rationalizing and simplifying yields $\frac{108+144i}{25}$.

When taken out of the complex plane, this corresponds to $(\frac{108}{25}, \frac{144}{25})$.

4.7 - Find all real x for which $x^5 + 5x^4 - 20x^2 - 2x + 14 = 0$.

ANSWER: -1, $\pm\sqrt{5 \pm \sqrt{2}} - 1$. We see no immediate tricks, so we depress the polynomial (removing the bx^{n-1} part) with the substitution $y = x + 1$ (in general the substitution is $y = x + \frac{b}{an}$ for degree n). This reduces the expression to $y^5 - 10y^3 + 23y = 0$, which is decidedly better. We immediatly get $y = 0$ as a solution, in addition to the malleable expression $y^4 - 10y^2 + 23 = 0$, which yields, $y^2 = \frac{10 \pm \sqrt{8}}{2} = 5 \pm \sqrt{2} \Longrightarrow y = \pm\sqrt{5 \pm \sqrt{2}}$. These 5 values of y give $x = -1, \pm\sqrt{5 \pm \sqrt{2}} - 1$.

4.8 - Four spheres $\Omega_1, \Omega_2, \Omega_3$, and Ω_4 are of radius 4, 6, 6, and 4 respectively. ξ is tetrahedron is formed with its vertices at the centers of each Ω_n. What is the volume of ξ?

ANSWER: $64\sqrt{3}$. Let A, B, C, and D correspond to the centers of Ω_1, Ω_2, Ω_3, and Ω_4 respectively. We have $AB = AC = DC = DB = 10, BC = 12$, and $DA = 8$. Let P be the midpoint of BC. Since ΔABC and ΔDBC are both isosceles, it is easy to find that $AE = DE = 8$. Because of the symmetric nature of this tetrahedron, the height to side AE in ΔAED is the same as the height from D of the tetrahedron. Finding the height to AE is ΔAED is easy because ΔAED is an equaliteral triangle. We find a height of $4\sqrt{3}$. We find the area of ΔABC to be 48 using AE and BC. Multiplying this by the height and then dividing by 3 produces the answer, $64\sqrt{3}$.