TJMC #5

NO CALCULATORS, 40 Minutes

5.1 - 2000! has 499 zeros at its end, and is a multiple of every positive integer up to and including 2002. It is not, however, a multiple of 2003. What is the remainder when it is divided by 2003?

5.2 - a_1, a_2, and a_3 are the first three terms of an arithmetic sequence of real numbers. Find all possible (a_1, a_2, a_3), given the following two properties relating a_1, a_2, and a_3:

$$i)\ (a_2)^2 = a_1 a_3 + 9$$
$$ii)\ (a_1)^4 + (a_2)^4 + (a_3)^4 = 4737$$

5.3 - Compute the value of the summation

$$\sum_{n=1}^{100} \frac{n^3 + 4n^2 + 3n + 2}{n^2 + 4n + 3}$$

5.4 - Five points $p_1, p_2, \ldots, p_5$ are selected on circle ω_1 such that $p_1p_2p_3p_4p_5$ is a regular pentagon. 5 chords are drawn connecting the points and forming a 5 pointed star shape. Circle ω_2 passes through the 5 vertices of the smaller pentagon formed by the 5 points of intersection between the chords. Find the ratio in terms of area between circles ω_1 and ω_2.

5.5 - Compute the value of the summation:

$$\sum_{i=1}^{100}\sum_{j=1}^{i}\sum_{k=1}^{j}\sum_{m=1}^{k} 1$$

5.6 - For how many integers $n > 1$ is the value of $\frac{1}{n}$ a repeating decimal that repeats immediately with 6 repeating digits? (Include n where $\frac{1}{n}$ is expressible as a repeating decimal with 1, 2, or 3 repeating digits.)

5.7 - $f(x)$ is the only polynomial of degree 2003 that passes through the points $(0,0), (1,0)$, $(2,0), (3,0), \ldots, (2001,0), (2002,0)$ and $(2003,1)$. Compute the numerical value of $f(2005)$.

5.8 - ABC is a scalene triangle in which $AB = 51$ and $AC = 30$. A point D is selected on BC such that $BD = 35$ and $DC = 28$. Let ω be the circumscribed circle of triangle ADC. γ_1 is a line that is tangent to ω. γ_2 is a line that is parallel to γ_1 that lies completely outside of ω. Ω is the torus that is formed by rotating ω about γ_2. Given that the distance between γ_1 and γ_2 is $\frac{31}{4}$, compute the volume of Ω.