# TJMC #5

NO CALCULATORS, 40 Minutes

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5.1 - 2000! has 499 zeros at its end, and is a multiple of every positive integer up to and including 2002. It is not, however, a multiple of 2003. What is the remainder when it is divided by 2003?

ANSWER: 1001. Working in $(mod\ 2003)$, we can obtain $2002! \equiv -1(mod\ 2003)$ by Wilson's theorem. We want to obtain 2000!, so we divide out 2001 * 2002 or 2 in $(mod\ 2003)$. Since -1 is 2002 in $(mod\ 2003)$, we have $2000! \equiv 1001(mod\ 2003)$.

5.2 - $a_1$, $a_2$, and $a_3$ are the first three terms of an arithmetic sequence of real numbers. Find all possible $(a_1, a_2, a_3)$, given the following two properties relating $a_1$, $a_2$, and $a_3$:

$$i)\ (a_2)^2 = a_1 a_3 + 9$$
$$ii)\ (a_1)^4 + (a_2)^4 + (a_3)^4 = 4737$$

ANSWER: $(2,5,8), (8,5,2), (-2,-5,-8), (-8,-5,-2)$. Because we are working with an arithmetic sequence, we can use the following substitutions: $a_1 = a - d$, $a_2 = a$, and $a_3 = a + d$. These produce $d^2 = 9$ when substituted into $i$). When used in $ii$), we obtain

$$(a-d)^4 + a^4 + (a+d)^4 = 4737$$
$$3a^4 + 12a^2d^2 + 2d^4 = 4737$$
$$3a^4 + 108a^2 - 4575 = 0$$
$$a^4 + 36a^2 - 1525 = 0$$
$$a^2 = \frac{-36 \pm \sqrt{36^2 - 4(1)(-1525)}}{2}$$
$$a^2 = \frac{-36 \pm 86}{2}$$
$$a^2 = 25 \implies a = \pm 5$$

Using the values of $a$ and $d$ produces the four answers.

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5.3 - Compute the value of the summation

$$\sum_{n=1}^{100} \frac{n^3+4n^2+3n+2}{n^2+4n+3}$$

ANSWER: $\frac{8843975}{1751}$ or $5050\frac{1425}{1751}$. The fraction can be quickly simplified to $n+\frac{2}{n^2+4n+3}$, and through partial fractions, $n+\frac{1}{n+1}-\frac{1}{n+3}$. The summation of $n$ from 1 to 100 is 5050. The other part of the summation telescopes down to $\frac{1}{2}+\frac{1}{3}-\frac{1}{102}-\frac{1}{103}$. Combining everything, we obtain $\frac{8843975}{1751}$.

5.4 - Five points $p_1, p_2, \ldots, p_5$ are selected on circle $\omega_1$ such that $p_1p_2p_3p_4p_5$ is a regular pentagon. 5 chords are drawn connecting the points and forming a 5 pointed star shape. Circle $\omega_2$ passes through the 5 vertices of the smaller pentagon formed by the 5 points of intersection between the chords. Find the ratio in terms of area between circles $\omega_1$ and $\omega_2$.

ANSWER: $\frac{7+3\sqrt{5}}{2}$. Let $P_1$ be the intersection of $p_1p_3$ and $p_2p_5$, and $P_2$ the intersection of $p_2p_4$ and $p_1p_3$. Because we are finding a ratio and no lengths are given, we can pick convenient lengths. Since everything is regular in nature, we let $p_3P_2 = P_2p_2 = p_2P_1 = P_1p_1 = 1$. We see that 5 angles congruent to $P_1p_2P_2$ have a sum of $180^o$. From this, we find that $mP_1p_2P_2 = 36^o$, and $mp_1P_1p_2 = 108^o$. Since both of the pentagons we compare the areas of are regular, there is a $K$ such that the area of the smaller circle is $K(P_1P_2)^2$ and the area of the larger circle is $K(p_1p_2)^2$. In finding the ratio, we see that $K$ is insignificant. The Law of Cosines from $P_1$ yields $(p_1p_2)^2 = 2-2\cos 108^o$, and from $p_2$ yields $(P_1P_2)^2 = 2-2\cos 36^o$. Using $\cos 36^o = \frac{1+\sqrt{5}}{4}$, we determine that $\cos 108^o = -\cos 72^o = -(2\cos 36^{o2}-1) = \frac{1-\sqrt{5}}{4}$. Finally, we have

$$\frac{2-2\cos 108^o}{2-2\cos 36^o} = \frac{1-\frac{1-\sqrt{5}}{4}}{1-\frac{1+\sqrt{5}}{4}} = \frac{4-(1-\sqrt{5})}{4-(1+\sqrt{5})} = \frac{3+\sqrt{5}}{3-\sqrt{5}} = \frac{7+3\sqrt{5}}{2}$$

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5.5 - Compute the value of the summation:

$$\sum_{i=1}^{100}\sum_{j=1}^{i}\sum_{k=1}^{j}\sum_{m=1}^{k} 1$$

ANSWER: 4421275. The summation can be simplified extensively by the application of a few basic summation formulas, after which it is easy to evaluate. Consider the following steps:

$$\sum_{i=1}^{100}\sum_{j=1}^{i}\sum_{k=1}^{j}\sum_{m=1}^{k} 1$$

$$\sum_{i=1}^{100}\sum_{j=1}^{i}\sum_{k=1}^{j} k$$

$$\sum_{i=1}^{100}\sum_{j=1}^{i}(\frac{j^2}{2}+\frac{j}{2})$$

$$\sum_{i=1}^{100}(\frac{2i^3+3i^2+i}{12}+\frac{i^2}{4}+\frac{i}{4})$$

$$\sum_{i=1}^{100}(\frac{i^3}{6}+\frac{i^2}{2}+\frac{i}{3})$$

$$\frac{((100)(101))^2}{24}+\frac{2(100)^3+3(100)^2+100}{12}+\frac{100(101)}{6}$$

$$4421275$$

5.6 - For how many integers $n > 1$ is the value of $\frac{1}{n}$ a repeating decimal that repeats immediately with 6 repeating digits? (Include $n$ where $\frac{1}{n}$ is expressible as a repeating decimal with 1, 2, or 3 repeating digits.)

ANSWER: 63. Consider a 6-digit repeating decimal $k$ of the form $0.\overline{abcdef}$. Multiplying by 1,000,000 gives $1000000k = abcdef.\overline{abcdef}$. We subtract $k$ and divide by 999,999 to obtain $k = \frac{abcdef}{999999}$. It is now clear that k is of the form $\frac{1}{n}$ if and only if $abcdef$ is a factor of 999999. We determine that $999,999 = 3^3 * 7 * 11 * 13 * 37$, and has $4*2*2*2*2$ or 64 factors. Since $n > 1$, $abcdef \neq 999,999$, and we have 63 possible $abcdef$.

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5.7 - $f(x)$ is the only polynomial of degree 2003 that passes through the points $(0,0), (1,0), (2,0), (3,0), \ldots, (2001,0), (2002,0)$ and $(2003,1)$. Compute the numerical value of $f(2005)$.

ANSWER: 2009010. Because all of the integers from 0 to 2002 are roots of $f(x)$, we express $f(x)$ as $f(x) = K(x)(x-1)(x-2)\cdots(x-2001)(x-2002)$. $f(x)$ also passes through $(2003,1)$, so $f(2003) = 1 = K(2003)(2002)(2001)\cdots(2)(1) = 2003! * K$. We have $K = \frac{1}{2003!}$.

This gives $f(2005) = \frac{2005!}{2003!*2} = \frac{2005*2004}{2} = 2009010$.

5.8 - $ABC$ is a scalene triangle in which $AB = 51$ and $AC = 30$. A point D is selected on BC such that $BD = 35$ and $DC = 28$. Let $\omega$ be the circumscribed circle of triangle ADC. $\gamma_1$ is a line that is tangent to $\omega$. $\gamma_2$ is a line that is parallel to $\gamma_1$ that lies completely outside of $\omega$. $\Omega$ is the torus that is formed by rotating $\omega$ about $\gamma_2$. Given that the distance between $\gamma_1$ and $\gamma_2$ is $\frac{31}{4}$, compute the volume of $\Omega$.

ANSWER: $12675\pi^2$. We first find $AD$. An application of Stuart's Theorem yields $51^2 * 28 + 30^2 * 35 = 35 * 63 * 28 + 63AD^2 \equiv 42588 = 63AD^2 \implies AD = 26$. Knowing the three sides of $\Delta$ADC, we apply Heron's formula to obtain its area. We obtain $S = 42, K = \sqrt{42(12)(14)(16)} = 336$. Using the property that $K = \frac{abc}{4R}$, we find that the radius of the circumscribed circle of $\Delta$ADC is $\frac{30*26*28}{4*336} = \frac{65}{4}$. We know that $\gamma_2$ is $\frac{31}{4}$ + the radius of $\omega$ or 24 away from the center of $\omega$. Finally, an application of the Theorem of Papus gives the volume of the torus formed by the rotation to be $2 * \pi * 24 * [\omega] = 48\pi * (\frac{65}{4})^2 * \pi = 12675\pi^2$.