TJMC #1 SOLUTIONS

NO CALCULATORS, 40 Minutes

1.1 - Compute A + B + C given the following equations:

$$2A + B + C = 8$$
$$A + 2B + C = 6 + \sqrt{3}$$
$$A + B + 2C = 10 - \sqrt{3}$$

ANSWER: 6. Adding the three equations produces $4(A + B + C) = 24$, from which it follows that $A + B + C = 6$.

1.2 - .Compute the positive integer remainder when $1^{2003} + 2^{2003} + ... + 2002^{2003} + 2003^{2003}$ is divided by 10.

ANSWER: 6. When working in (mod 10), all of the bases are reduced to 1, 2, $\cdots$, 9, or 0. We have 200 sets of $\{0 \text{ - } 9\}^{2003}$, so all of those cancel when divided by 10. We are left with finding $1^{2003} + 2^{2003} + 3^{2003}$ (mod 10). Looking at the patterns formed in the units place, we conclude that this is $1 + 8 + 7 \equiv 6$ (mod 10).

1.3 - $f(x)$ is a cubic function in x such that its three roots are 1, 2002, and 2003. Compute the value of $\frac{f(2000)}{f(2001)}$.

ANSWER: $\frac{5997}{2000}$. From the given roots, $f(x) = (x - 1)(x - 2002)(x - 2003)$. The requested expression becomes: $\frac{(2000-1)(2000-2002)(2000-2003)}{(2001-1)(2001-2002)(2001-2003)}$. This can be evaluated easily: $\frac{(1999)(-2)(-3)}{(2000)(-1)(-2)} = \frac{5997}{2000}$.

1.4 - ABCDEF is a regular hexagon of side length $\frac{2002}{2003}$. Compute the value of $(\frac{AC}{AD})^2$.

ANSWER: $\frac{3}{4}$. Since ΔACD is a $30-60-90$ right triangle with right angle C, we find that we are asked to compute the square of its longer leg divided by the square of its hypotenuse. This is $(\frac{\sqrt{3}}{2})^2 = \frac{3}{4}$.

1.5 - Simplify the following expression as far as possible, while keeping your answer exact:

$$\sqrt[3]{\sqrt{18+\sqrt{128}}+\sqrt{18-\sqrt{128}}}+\sqrt[3]{\sqrt{18+\sqrt{128}}-\sqrt{18-\sqrt{128}}}$$

ANSWER: $2+\sqrt{2}$. Consider the following algebra:

$$\begin{aligned}
x &= \sqrt{a+\sqrt{b}}+\sqrt{a-\sqrt{b}} \\
x^2 &= a+\sqrt{b}+2\sqrt{a^2-b}+a-\sqrt{b} \\
x &= \sqrt{2a+2\sqrt{a^2-b}}
\end{aligned}$$

$$\begin{aligned}
y &= \sqrt{a+\sqrt{b}}-\sqrt{a-\sqrt{b}} \\
y^2 &= a+\sqrt{b}-2\sqrt{a^2-b}+a-\sqrt{b} \\
y &= \sqrt{2a-2\sqrt{a^2-b}}
\end{aligned}$$

The substitution a = 18, b = 128 reduces the original expression to $\sqrt[3]{8}+\sqrt[3]{2\sqrt{2}} = 2+\sqrt{2}$.

1.6 - $f(x)$ is a monic, cubic function with three distinct roots. Let the a, b, and c be these roots. Compute $f(1)$ given the three following equations relating a, b, and c.

$$\begin{aligned}
a+b+c &= 1 \\
ab+ac+bc &= 1002 \\
abc &= 2003
\end{aligned}$$

ANSWER: -1001. If the three roots of f are a, b, and c, then we have $f(x) = (x-a)(x-b)(x-c)$. We then consider the following expansion and substitution:

$$\begin{aligned}
f(1) = (1-a)(1-b)(1-c) &= 1-(a+b+c)+(ab+bc+ac)-abc \\
&= 1-1+1002-2003 \\
&= -1001
\end{aligned}$$

1.7 - Find all order pairs (a, b) for real a and b such that

$$a + b - 10\sqrt{a+b} + 25 = 9$$
$$a - b - 10\sqrt{a-b} + 49 = 25$$

ANSWER: $(10, -6), (40, 24), (20, -16), (50, 14)$. We consider the substitutions $x = \sqrt{a-b}$ and $y = \sqrt{a+b}$. We obtain the following:

$$x^2 - 10x + 24 = 0 \Longrightarrow x = 4 \text{ or } 6$$
$$y^2 - 10y + 16 = 0 \Longrightarrow y = 2 \text{ or } 8$$

Solving simultaneously for a and b in each of the four possible (x, y) produces the desired order pairs of (a, b).

1.8 - ABCD is a convex quadrilateral inscribed in circle O such that AB = 5, BC = 8, and CD = 7. Chord AD is extended beyond A to point P such that AP = 2. A tangent to circle O from P is of length 4. Compute the numerical value of BD^2.

ANSWER: $\frac{3403}{43}$. Through power of a point from P, we find that $PD(PD + DA) = PC^2$, from which we can determine that $DA = 6$. We let $mDAB = \gamma$, which also implies that $mDCB = \pi - \gamma$. Two applications of the Law of Cosines from A and C yield the following:

$$BD^2 = 5^2 + 6^2 - 2 * 5 * 6 \cos\gamma = 61 - 60 \cos\gamma$$
$$BD^2 = 7^2 + 8^2 - 2 * 7 * 8 \cos(\pi - \gamma) = 113 - 112 \cos(\pi - \gamma)$$

We now solve for $\cos\gamma$, and then use this value to compute BD^2:

$$61 - 60 \cos\gamma = 113 - 112 \cos(\pi - \gamma)$$
$$-52 = 172 \cos\gamma$$
$$\cos\gamma = \frac{-13}{43}$$

$$BD^2 = 61 - 60 * \frac{-13}{43}$$
$$BD^2 = \frac{3403}{43}$$