TJ USAMO Practice 1 Solutions

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October 24, 2003

1. (MOP 2003) Let ABC be a triangle with $AB \neq AC$. Let D be the foot of the altitude from A to $\overline{BC}$. Let P be a point on $\overline{AD}$ and let E and F be the intersections of $\overrightarrow{BP}$ and $\overrightarrow{CP}$ with $\overline{AC}$ and $\overline{AB}$ respectively. Show that if $CEFB$ is a cyclic quadrilateral, then P is the orthcenter of triangle ABC.

   Solution

   We assume $AC < AB$.[1]

   If $CEFB$ is cyclic,[2] then $\angle ECF \cong \angle EBF$ and $\angle CEB \cong \angle CFB$. We reflect C over $\overline{AD}$ to point C', giving $\angle PCC' \cong \angle CC'P$, and $\angle PC'A \cong \angle ACF \cong \angle EBA \implies$ quadrilateral $BC'PA$ is cyclic $\implies \angle C'AP \cong \angle C'BP$.

   We equate the sums of the angles in triangles CEB and $C'DA$, and have:

$$\begin{aligned} 180^o &= m\angle CEB + m\angle EBC + m\angle BCP + m\angle PCE & (1)\\ &= m\angle C'DA + m\angle DAC' + m\angle AC'P + m\angle PC'D & (2)\end{aligned}$$

   Cancelling pairwise using our congruences, we are left with $m\angle CEB = m\angle C'DA \implies \angle CEB \cong \angle C'DA \cong \angle CFB$. Because $\angle CD'A$ is a right angle, $\overline{BE}$ and $\overline{CF}$ are altitudes, and it follows that P is the orthocenter of ABC.

2. (Romania 1997) Find all continuous functions $f : \mathbb{R} \to [0, \infty)$ such that $\forall x, y \in \mathbb{R}$,[3]

$$f(x^2 + y^2) = f(x^2 - y^2) + f(2xy)$$

   Solution

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[1] We are allowed to assume this because we are given $AC \neq AB$ and the solution can easily be shown to work in the opposite case $AC > AB$.

[2] Cyclic quadrilaterals give many congruences. If you do not follow all of the following arguments, then see the lecture on cyclic quadrilaterals.

[3] That is, f is a function that accepts any real number and returns a non-negative real number, and also for any real numbers x and y, satisfies...

First, substitute $x = -x$ to show that $f(x^2+y^2) = f(x^2-y^2)+f(-2xy)$. Subtracting this from the given, and using the fact that x and y are unrestricted in the result, establishes that $f(x) = f(-x)$.

We define $g(x) : [0, \infty) \to [0, \infty) = f(\sqrt{x})$. Then, from the given equation, we have $g(x^4+2x^2y^2+y^4) = g(x^4-2x^2y^2+y^4)+g(4x^2y^2)$. We will argue that the two expressions $x^4 - 2x^2y^2 + y^4$ and $4x^2y^2$ can assume any two positive values.

Since both are positive, we can take the square root of each, so the desired result is equivalent to $x^2 - y^2 = a$ and $2xy = b$ having solutions for any a and b. Substituting the expression for y in terms of x and b derived from the right equation yields $x^4 - ax^2 - \frac{b^2}{4} = 0$. The discriminant of this is greater than 0, so it has a solution, so $x^4 + y^4 - 2x^2y^2$ and $4x^2y^2$ can be any two positive values.

So we have $g(a)+g(b) = g(a+b)$ for any $a, b \in \mathbb{R}^+$.[4] Set $g(1) = k$. From the additivity, we can "break up" any $g(x)$ so $pg(x) = g(px)$, $\forall p \in \mathbb{Z}^+$.[5] Setting $x = \frac{x}{p}$ and $p = q$, we can rearrange this to $g(\frac{x}{q}) = \frac{g(x)}{q}$ $\forall q \in \mathbb{Z}^+$. Therefore, $g(\frac{p}{q}) = k\frac{p}{q}$, for any positive integers p and q.

Because f is continuous, g is continuous. Because the rationals are ‡dense in the reals and g is continuous, and because $g(x) = kx$ for any positive rational x, $g(x) = kx$ for any positive real x.

Because $g(x) = f(\sqrt(x))$, $g(x^2) = f(x)$, which means $f(x) = kx^2$ for positive x. Because setting $x = y = 0$ yields $f(0) = 0$, and because $f(x) = f(-x)$, we have $f(x) = kx^2$ for all real x.

3. (USAMO 1974) Prove that if a, b, and c are positive real numbers, then

$$a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}$$

Solution

Cubing each side and dividing by $a^a b^b c^c$ shows the given to be equivalent to

$$a^{2a} b^{2b} c^{2c} \geq a^{b+c} b^{a+c} c^{a+b}$$

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[4] For any positive real numbers a, b.

[5] For any positive integer p.

Lemma:[6] $\forall a,\, b \in \mathbb{R}^+$, $a^a b^b \geq a^b b^a$. We assume, WLOG,[7] $a \geq b$. If we divide each side by $a^b b^b$, we are left with $a^{a-b} \geq b^{a-b}$, which true since $a \geq b$. Finally,

$$a^{2a} b^{2b} c^{2c} = a^a b^b a^a c^c b^b c^c \geq a^{b+c} b^{a+c} c^{a+b}$$

which is true by our lemma.

4. (USAMO 1976) If $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ are all polynomials such that

$$P(x^5) + xQ(x^5) + x^2 R(x^5) = (x^4 + x^3 + x^2 + x + 1)S(x)$$

prove that $x - 1$ is a factor of $P(x)$.

Solution

Let $[x^n]P$ denote the coefficient of the x^n term of $P(x)$.[8]

Define $T(x) = P(x^5) + xQ(x^5) + x^2R(x^5) = (x^4 + x^3 + x^2 + x + 1)S(x)$. Then we have:

$$\begin{aligned}
[1]P &= [1]T = [1]S \\
0 &= [x^4]T = [1]S + [x]S + [x^2]S + [x^3]S + [x^4]S \\
[x]P &= [x^5]T = [x]S + [x^2]S + [x^3]S + [x^4]S + [x^5]S \\
[x]P &= [x^5]S - [1]S \\
i : [1]P + [x]P &= [x^5]S
\end{aligned}$$

Similarly,

$$\begin{aligned}
0 &= [x^{5n+4}]T = [x^{5n}]S + [x^{5n+1}]S + [x^{5n+2}]S + [x^{5n+3}]S + [x^{5n+4}]S \\
[x^{n+1}]P &= [x^{5n+5}]T = [x^{5n+1}]S + [x^{5n+2}]S + [x^{5n+3}]S + [x^{5n+4}]S + [x^{5n+5}]S \\
ii : [x^{5n}]S + [x^{n+1}]P &= [x^{5n+5}]S
\end{aligned}$$

By i and ii by induction,[9] we have $[x^{5n}]S = [1]P + [x]P + \cdots + [x^n]P$.

If $5n$ is greater than the degree of S, then $[x^{5n}]S = 0$. If n is at least the degree of P, then $[1]P + [x]P + \cdots + [x^n]P = [1]P + 1[x]P + \cdots + 1^n[x^n]P = P(1)$. So if n is large enough, $P(1) = 0$, so $x - 1$ divides P.

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[6] A lemma is an intermediate proof. It is the formal way to show a simpler identity that applies to the main problem.

[7] Without Loss Of Generality.

[8] You can define any expressions you like as long as you define them before you use them. Here, this shorthand notation simplifies referring to a coefficient of a polynomial.

[9] Induction is a technique for generalizing a result. The outline of induction is, if we have a base case, and show that case n implies case $n + 1$, then the result must be true for all n. Usually use of induction will require more proof, but in this case it is reasonable to omit details because they are simple.

5. (USAMO 1981) If A, B, and C are the angles of a triangle, prove that

$$-2 \leq \sin 3A + \sin 3B + \sin 3C \leq \frac{3\sqrt{3}}{2}$$

and determine when equality holds.

Solution

First, we will show the left side of the inequality. Equality holds where $A = B = 90^o$ and $C = 0^o$. Since $-1 \leq \sin\theta \leq 1$, any lower value requires that all three sines must be negative. For $\sin\theta < 0$, where θ is positive, we must have $\theta > 180^o$, but if $3A, 3B, 3C > 180^o$, one third of their sum exceeds 180^o, so we have a contradiction. So the left holds.

We set $a = 3A, b = 3B, c = 3C$, so we have $a + b + c = 540^o$. First note that $A = B = 20^o$, $C = 140^o$ is an equality case for the right hand side. If only two of the sines are positive, the maximum possible value is 2, but $2 < \frac{3\sqrt{3}}{2}$, so all three sines must be positive. That is, $a, b, c \in (0^o, 180^o) \cup (360^o, 540^o)$. Since $a + b + c = 540^o$, exactly one of a, b, and c exceeds 360^o. WLOG, this is c.

Set $x = a, y = b, z = c - 360^o$, with $0^o \leq x \leq y \leq z \leq 180^o$. We know that $x + y + z = 180^o$, and we have $\sin x + \sin y + \sin z = \sin a + \sin b + \sin c$, so if we maximize the left expression, then we have also maximized the right expression.

Because the sine function is concave[10] on the interval $(0^o, 180^o)$, we can apply Jensen's inequality,[11] which gives $\frac{\sin x + \sin y + \sin z}{3} \leq \sin\frac{x+y+z}{3} = \sin 60^o = \frac{\sqrt{3}}{2}$.

It follows that $\sin 3A + \sin 3B + \sin 3C = \sin a + \sin b + \sin c = \sin x + \sin y + \sin z \leq \frac{3\sqrt{3}}{2}$.

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[10] Calling a function f concave on an interval means that the slope of f is decreasing as we move from left to right. The term convex applies in the opposite manner.

[11] This is an extremely powerful inequality. It relates the expressions $\frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}$ and $f(\frac{x_1+x_2+\cdots+x_n}{n})$, where x_i are all numbers in a given interval. If f is concave in this interval, then the first expression is less than or equal to the second expression, and if f is convex on the interval, the sign flips.