

TJ USAMO Practice 1

VMT Math Team

October 24, 2003

1. (MOP 2003) Let ABC be a triangle with $AB \neq AC$. Let D be the foot of the altitude from A to \overline{BC} . Let P be a point on \overline{AD} and let E and F be the intersections of \overline{BP} and \overline{CP} with \overline{AC} and \overline{AB} respectively. Show that if $CEFB$ is a cyclic quadrilateral, then P is the orthcenter of triangle ABC .

2. (Romania 1997) Find all continuous functions $f : \mathbb{R} \rightarrow [0, \infty)$ such that for all $x, y \in \mathbb{R}$,

$$f(x^2 + y^2) = f(x^2 - y^2) + f(2xy)$$

3. (USAMO 1974) Prove that if a, b , and c are positive real numbers, then

$$a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}$$

4. (USAMO 1976) If $P(x), Q(x), R(x)$, and $S(x)$ are all polynomials such that

$$P(x^5) + xQ(x^5) + x^2R(x^5) = (x^4 + x^3 + x^2 + x + 1)S(x)$$

prove that $x - 1$ is a factor of $P(x)$.

5. (USAMO 1981) If A, B , and C are the angles of a triangle, prove that

$$-2 \leq \sin 3A + \sin 3B + \sin 3C \leq \frac{3\sqrt{3}}{2}$$

and determine when equality holds.