TJ USAMO Practice 10

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1. Determine the minimum positive integer n for which the diophantine equation

$$x_1^6 + x_2^6 + \cdots + x_n^6 = 7,000,064$$

has a solution.

Solution

We take modulo 9: $0^6 = 0 \to 0, 1^6 = 1 \to 1, 2^6 = 64 \to 1, 3^6 = 729 \to 0, 4^6 = 4096 \to 1, 5^6 \equiv (-4)^6 \to 1, 6^6 \equiv (-3)^6 \to 0, 7^6 \equiv (-2)^6 \to 1, 8^6 \equiv (-1)^6 \to 1$. When divided by 9, the number 7,000,064 leaves a remainder of 8. Therefore the equation in mod 9 is equivalent to:

$$r_1 + r_2 + \cdots + r_n = 8$$

Where each of the r_i is restricted to 0 or 1. It follows that n is at least 8, and the solution { 10, 10, 10, 10, 10, 10, 10, 2 } confirms 8 as the minimum n.

2. Prove that, for all positive real numbers a, b, c,

$$(a^3 + b^3 + abc)^{-1} + (b^3 + c^3 + abc)^{-1} + (c^3 + a^3 + abc)^{-1} \leq (abc)^{-1}.$$

Solution

We first prove that $a^3 + b^3 \geq ab(a + b)$ for positive reals a and b. This follows from:

$$\begin{aligned} (a-b)^2 &\geq 0 \\ \Longrightarrow a^2 - ab + b^2 &\geq ab \\ \Longrightarrow a^3 + b^3 = (a+b)(a^2 - ab + b^2) &\geq ab(a+b) \end{aligned}$$

Therefore:

$$\frac{1}{a^3+b^3+abc} + \frac{1}{b^3+c^3+abc} + \frac{1}{c^3+a^3+abc} \leq \frac{1}{ab(a+b+c)} + \frac{1}{bc(a+b+c)} + \frac{1}{ca(a+b+c)}$$

The stronger inequality

$$\frac{1}{ab(a+b+c)} + \frac{1}{bc(a+b+c)} + \frac{1}{ca(a+b+c)} \leq \frac{1}{abc}$$

is true because when both sides are multiplied by abc, the result is $1 \leq 1$. Q.E.D.

3. Find all positive integers n for which

$$\frac{3n+200}{10n-1}$$

is reducible.

Solution

Let p be a factor that can be extracted from both the numerator and denominator. Therefore $p|3n+200$ and $p|10n-1 \Longrightarrow p|q(3n+200)+r(10n-1)$, for any integers q and r. Setting $q=10$ and $r=-3$ yields $p|2003$. Because 2003 is prime, p must be 1 or 2003. Since extracting a factor of 1 does not reduce the fraction, it follows that p = 2003.

Therefore $3n+200 \equiv 0 \pmod{2003} \iff n \equiv 601 \pmod{2003}$ and $10n-1 \equiv 0 \pmod{2003} \iff n \equiv 601 \pmod{2003}$. The irreducible values where the numerator and denominator are 0 occur at $n = \frac{-200}{3}$ and $n = \frac{1}{10}$ respectively. It follows that the given fraction is reducible $\forall n = 601 + 2003k : k \in \mathbb{Z}^{*+}$

4. (MOP 03) On a table lies a point X and several face clocks, not necessarily identical. Each face clock consists of a fixed center, and two hands (a minute hand and an hour hand) of equal length. (The hands rotate around the center at a fixed rate; each hour, a minute hand makes a complete revolution while an hour hand completes 1/12 of a revolution.) It is known that at some point, the following two quantities are distinct:

- the sum of the distances between X and the end of each minute hand, and
- the sum of the distances between X and the end of each hour hand.

Prove that at some moment, the former sum is greater than the latter sum.

Solution

Let $m(t)$ represent the former sum, $h(t)$ the latter, and let I be any continuous 12-hour interval of time. Suppose there are n clocks and let $m_i(t)$ represent the distance from the tip of the minute hand on the ith clock to X, where $1 \leq n$. Define $h_i(t)$ analogously. The functions m and h are given by $m(t) = m_1(t) + \cdots + m_n(t)$ and

$h(t) = h_1(t) + \cdots + h_n(t)$.

Consider the graphs of $m_i(t)$ and $h_i(t)$ over $t \in I$. Because the hands rotate at a constant rate, $m_i(t)$ and $h_i(t)$ are continuous graphs. By simple symmetry, the average value of $m_i(t)$ is the same as the average value of $h_i(t)$. It follows that the areas under the graphs of $m_i(t)$ and $h_i(t)$ over $t \in I$ are identical. Summing this result over i yields the fact that the areas under the graphs of $m(t)$ and $h(t)$ where $t \in I$ are identical.

We are given that there exists a time t_1 such that $m(t_1) \neq h(t_1)$. If $m(t_1) > h(t_1)$, then we are done. Otherwise, $m(t_1) < h(t_1)$. Let T be a continuous half-day that contains t_1. Because the functions $m(t)$ and $h(t)$ are continuous, there is a region directly above $m(t_1)$ that is under the graph of $h(t)$ but above the graph of $m(t)$. But it was established earlier that the areas under the graphs of $m(t)$ and $h(t)$ over any 12-hour time period are the same. Therefore, there must be some region above the graph of $h(t)$ and below the graph of $m(t)$ over $t \in T$. Therefore there exists some time $t_2 \in T : m(t_2) > h(t_2)$. Q.E.D.

5. (USAMO 91) Let X be a point on side BC of triangle ABC. Let Y be the intersection of AX and the common tangent (other than BC) of the incircles of ABX and ACX. Show that the locus of Y as X varies is the arc of a circle.

Solution

We will argue that $AY = \frac{AB+AC-BC}{2}$, so that the locus of Y as X varies is actually the arc of a circle centered at A.

Let R and S denote the intersections of the common tangent with the incircles of $\triangle ABX$ and $\triangle ACX$ respectively, let $\overline{AX}$ meet them at P and Q respectively, and let $\overline{BC}$ meet them at U and V respectively. Let $\overline{AB}$ be tangent to the incircle of $\triangle ABX$ at K and $\overline{AC}$ be tangent to the incircle of $\triangle ACX$ at L.

We have $AY = AQ - QY$ and $AY = AP - PY$. Adding, $2AY = AP + AQ - (YQ + YP) = AP + AQ - (YS + YR) = AP + AQ - RS$. Because the lengths of common external tangents are equal, $RS = UV$ so that $2AY = AP + AQ - UV$. We note that $AP = AK = AB - BK = AB - BU$ and $AQ = AL = AC - CL = AC - CV$. Finally, $2AY = AB + AC - BU - UV - CV = AB + AC - BC$, a constant independent of X. Q.E.D.