TJ USAMO Practice 10

VMT Math Team

January 12, 2004

1. Determine the minimum positive integer n for which the diophantine equation

$$x_1^6 + x_2^6 + \cdots + x_n^6 = 7,000,064$$

has a solution.

2. Prove that, for all positive real numbers a, b, c,

$$(a^3 + b^3 + abc)^{-1} + (b^3 + c^3 + abc)^{-1} + (c^3 + a^3 + abc)^{-1} \leq (abc)^{-1}.$$

3. Find all positive integers n for which

$$\frac{3n + 200}{10n - 1}$$

is reducible.

4. (MOP 03) On a table lies a point X and several face clocks, not necessarily identical. Each face clock consists of a fixed center, and two hands (a minute hand and an hour hand) of equal length. (The hands rotate around the center at a fixed rate; each hour, a minute hand makes a complete revolution while an hour hand completes 1/12 of a revolution.) It is known that at some point, the following two quantities are distinct:

   - the sum of the distances between X and the end of each minute hand, and
   - the sum of the distances between X and the end of each hour hand.

   Prove that at some moment, the former sum is greater than the latter sum.

5. (USAMO 91) Let X be a point on side BC of triangle ABC. Let Y be the intersection of AX and the common tangent (other than BC) of the incircles of ABX and ACX. Show that the locus of Y as X varies is the arc of a circle.