TJ USAMO Practice 11

VMT Math Team

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1. Three piles of stones contain 100, 2000, and 2004 stones respectively. If we are allowed to pick two of the piles and move one stone from each to the third pile, is it possible to eventually wind up with three piles of 1368 stones?

2. Show that given any polygon, three points on it can be selected that form an equilateral triangle.

3. Show that, for positive reals a, b, and c such that $a + b + c = 1$,

$$\sqrt[3]{9 + \frac{8}{3abc}} \geq \sqrt[3]{\frac{1}{a}} + \sqrt[3]{\frac{1}{b}} + \sqrt[3]{\frac{1}{c}}$$

4. Prove that there exists an integer m such that prime p divides $m^2 + 4m + 15$ iff there exists an integer n such that p divides $3n^2 + 8n + 20$.

5. Determine all pairs of positive integers (m, n) for which $2^m + 3^n$ is a perfect square.