

TJ USAMO Practice 13

VMT Math Team

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1. (MOP 03) Let \mathbb{N} denote the set of positive integers, and let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function such that $f(m) > f(n)$ for all $m > n$. For positive integer m , let $g(m)$ denote the number of positive integers n such that $f(n) < m$. Express

$$\sum_{i=1}^{f(n)} g(i) + \sum_{k=1}^n f(k)$$

in closed form.

2. (USAMO 89) An acute-angled triangle has unequal sides. Show that the line through the circumcenter and incenter intersects the longest side and the shortest side.
3. (Zvezda98) Prove for all nonnegative numbers a, b, c :

$$\frac{(a + b + c)^2}{3} \geq a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}$$

4. (IMO 96) Let S be the set of non-negative integers. Find all functions $f : S \rightarrow S$ such that $f(m + f(n)) = f(f(m)) + f(n)$ for all $m, n \in S$.
5. (USAMO 92) A complex polynomial has degree 1992 and distinct zeros. Show that we can find complex numbers z_n , such that if $p_1(z) = z - z_1$ and $p_n(z) = p_{n-1}(z)^2 - z_n$, then the polynomial divides $p_{1992}(z)$.