TJ USAMO Practice 18 - Functional Equations

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Put bluntly, functional equations are not easy. From clever tricks to general theorems, their solutions are highly varied. A few tools are listed here:

- (Cauchy) All functions $f : \mathbb{Q} \to \mathbb{Q}$ such that the *Cauchy Equation* $f(x+y) = f(x) + f(y)$ are given by $f(x) = cx$, where $c = f(1)$.
- (Density of Rational Numbers) For any real number x, there exist rational numbers p and q such that $p < x < q$ and $|x-p|, |x-q| < \epsilon$ for any positive ϵ.
- (Sandwich Theorem) If f is monotone (increasing or decreasing) and $\lim_{x \in \mathbb{D} \to c^+} f(x) = y = \lim_{x \in \mathbb{D} \to c^-}$ where $\mathbb{D}$ is a dense subset of the domain of $f(x)$, then $f(c) = y$.
- (Fixed points) Consider all values x in the domain of f such that $f(x) = x$. Such values are called *fixed points*.

The following problems employ these techniques:

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $x^2 f(x) + f(1-x) = 2x - x^4$ for all $x \in \mathbb{R}$.
2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(x)$ for all $x, y \in \mathbb{R}$ and $f(x) \geq 0$ for $x \geq 0$.
3. (IMO 2002) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

   $$(f(x) + f(z))(f(y) + f(t)) = f(xy - zt) + f(xt + yz)$$

   for any $x, y, z, t \in \mathbb{R}$.
4. (IMO 1983) Determine all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(xf(y)) = yf(x)$ for all $x, y \in \mathbb{R}^+$ and as $x \to \infty$, $f(x) \to 0$.
5. (IMO 1996) Find all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that

   $$f(m + f(n)) = f(f(m)) + f(n)$$

   for all $m, n \in \mathbb{N}_0$.