TJ USAMO Practice 2 Solutions

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1. (USAMO 1973) Let X_n and Y_n denote two sequences of integers defined as follows:

$$X_0 = 1, X_1 = 1, X_{n+1} = X_n + 2X_{n-1} \quad (n = 1,\ 2,\ 3,\ \ldots)$$
$$Y_0 = 1, Y_1 = 7, Y_{n+1} = 2Y_n + 3Y_{n-1} \quad (n = 1,\ 2,\ 3,\ \ldots)$$

   Prove that, except for Y_0, there is no Y_i that appears in the sequence X_j.

   Solution

   We consider the sequences in modulo 8.[1]

   We have the sequence $X_n = 1, 1, 3, 5, 11, 21, 43, \ldots$ which, when each term is taken modulo 8, becomes $1, 1, 3, 5, 3, 5, \ldots$. Because a given term is defined in terms of the previous two terms, and the two consecutive terms 3 and 5 give rise to another 3 and 5, we know that the pattern $3, 5, 3, 5, \ldots$ continues forever.

   Similarly, Y_n becomes $1, 7, 1, 7, \ldots$, where the pattern $1, 7, 1, 7, \ldots$ continues infinitely.

   If $X_i = Y_j$ for some i and j, then the residues modulo 8 must also be equal. The only residues that appear in both sequences are the 1's. Notably, the only number with residue 1 in modulo 8 that appears in X_n is 1. This appears only once in Y_n because it is clear that Y_j increases as j increases.

   It follows that the only term in Y_n that also appears in X_n is Y_0, and we are done.

2. (USAMO 1978) Nine mathematicians meet at an international conference and discover that among any three of them, at least two speak a common language. If each of the mathematicians speaks at most three languages, prove that there are at least three of

[1] Whenever we have sequence problem, in particular when we are asked to show that a certain term only appears a finite number of times, we immediately consider modulos.

the mathematicians who can speak the same language.

Solution

We will prove by contradiction; that if no three mathematicians speak the same language, then some given must be false.

Let M_1 be a mathematician, that speaks $i \leq 3$ languages. Because no three mathematicians speak the same language, then at most two speak the same language, which means that at most three other mathematicians speak a common language with $M_1 \Longrightarrow$ at least 5 mathematicians share no language with M_1. Let S be the set of mathematicians who do not share a language with M_1.

We have $M_2 \in S$. By similar reasoning, at most three mathematicians in S share a language with M_2. This means that at least one other mathematician in S cannot speak with M_2. We denote this subset of S with the letter T.

We have $M_3 \in T$. M_1 and M_2 do not share a language, nor do M_1 and M_3, since $M_3 \in T \subseteq S$. But M_2 and M_3 do not share a language either, a contradiction to the first given.

We assumed no three mathematicians speak the same language, but this must be false.

3. (USAMO 1979) Given three identical n-faced dice whose corresponding faces are indentically numbered with arbitrary integers, prove that if they are tossed at random, the probability that the sum of the bottom three face numbers is divisible by three is greater than or equal to $\frac{1}{4}$.

Solution

Let us say that on any given die, the probability that it yields an integer that is 0 modulo 3 is p, that the chance for 1 modulo 3 is q, and that the chance for 2 modulo 3 is q. Then we have $p+q+r=1$, because a number must be 0, 1, or 2 modulo 3.

The chance that the sum of three die is a multiple of 3 is $p^3+q^3+r^3+6pqr$, because to get 0 modulo 3 we must have three of the same residue or any of the six permutations of 0, 1, and 2. The problem becomes equivalent to showing:

$$p+q+r=1 \Longrightarrow p^3+q^3+r^3+6pqr \geq \frac{1}{4}$$

We factor extensively[2]:

$$\begin{aligned} p^3+q^3+r^3+6pqr &= (p+q+r)(p^2-pq+q^2-qr+r^2-rp)+9pqr & (1)\\ &= p^2-pq+q^2-qr+r^2-rp+9pqr & (2)\\ &= (p+q+r)^2-3(pq+qr+rp)+9pqr & (3)\\ &= 1-3(pq+qr+rp)+9pqr & (4) \end{aligned}$$

So, where $p+q+r=1$, we have:

$$p^3+q^3+r^3+6pqr = 1-3(pq+qr+rp)+9pqr \geq \frac{1}{4} \iff pq+qr+rp-3pqr \leq \frac{1}{4}$$

WLOG, $p \leq q \leq r$, which implies $\frac{1}{3} \leq r \Longrightarrow 1-3r \leq 0$. More algebra yields:

$$pq+qr+rp-3pqr = pq+r(p+q)-3pqr = r(1-r)+pq(1-3r) \leq r(1-r)$$

We apply AM-GM to obtain the identity $\frac{r+(1-r)}{2} \geq \sqrt{r(1-r)} \iff \frac{1}{4} \geq r(1-r)$. Therefore,

$$pq+qr+rp-3pqr \leq \frac{1}{4} \Longrightarrow p^3+q^3+r^3+6pqr \geq \frac{1}{4}$$

and we are done.

4. (USAMO 1981) If x is a positive real number, and n is a positive integer, prove that

$$[nx] \geq \frac{[x]}{1}+\frac{[2x]}{2}+\frac{[3x]}{3}+\cdots+\frac{[nx]}{n}$$

where [t] denotes the greatest integer less than or equal to t.

Solution

Note that $nx = \frac{x}{1}+\frac{2x}{2}+\frac{3x}{3}+\cdots+\frac{nx}{n}$ and if $[nx] \geq \frac{[x]}{1}+\frac{[2x]}{2}+\frac{[3x]}{3}+\cdots+\frac{[nx]}{n}$ then $nx-[nx] \leq \frac{x-[x]}{1}+\frac{2x-[2x]}{2}+\frac{3x-[3x]}{3}+\cdots+\frac{nx-[nx]}{n}$.

Let $[[x]] = x-[x]$ be the fractional part of x. Since $nx = \frac{x}{1}+\frac{2x}{2}+\frac{3x}{3}+\cdots+\frac{nx}{n}$, we wish to show that

$$[[nx]] \leq \frac{[[x]]}{1}+\frac{[[2x]]}{2}+\frac{[[3x]]}{3}+\cdots+\frac{[[nx]]}{n}$$

Both sides of this inequality are of slope n with all discontinuities jumping down. The left jumps down at $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \cdots$. If the inequality is violated at some r, then it is violated up to the next multiple of $\frac{1}{n}$. Suppose that the inequality is violated at r, and that

[2] We do this because it is obvious that the inequality is true for large, unrestricted p, q, and r, so we want to assert the restriction $p+q+r=1$ as much as possible

the next multiple of $\frac{1}{n}$ is $\frac{k}{n}$. Let $r = \frac{k}{n} - \epsilon$, where $0 \leq \epsilon < \frac{1}{n}$. Let $(n,\, k) = d$ and $\frac{n}{d} = c$. Modulo n, the numbers $k,\, 2k, \ldots,\, ck$ are permutations of $0,\, d,\, 2d, \ldots,\, (c-1)d$. Hence, $\frac{[[r]]}{1},\, \frac{[[2r]]}{2},\, \frac{[[3r]]}{3}, \ldots,\, \frac{[[cr]]}{c}$ are within ϵ of some rearrangement of $\frac{d}{n},\, \frac{2d}{n}, \ldots,\, \frac{cd}{n}$. Therefore,

$$\frac{[[r]]}{1}+\frac{[[2r]]}{2}+\frac{[[3r]]}{3}+\cdots+\frac{[[cr]]}{c} \geq \frac{d}{n} = \frac{2d}{n}+\frac{3d}{n}+\ldots+\frac{cd}{n}-c\epsilon \geq \frac{cd}{n}-n\epsilon = 1-n\epsilon = [[k-n\epsilon]] = [[nr]]$$

5. (USAMO 1982) Let $S_r = x^r + y^r + z^r$ with $x, y, z \in \mathbb{R}$. It is known that if $S_1 = 0$,

$$\frac{S_{m+n}}{m+n} = \frac{S_m}{m}\frac{S_n}{n}$$

For $(m, n) = (2, 3), (3, 2), (2, 5)$, or $(5, 2)$. Determine all other pairs of integers (m, n), if any, so that the above identity holds for all real numbers x, y, z such that $x+y+z = 0$.

Solution

We will show that the equation does not hold for any other pairs of integers.[3]

First, $m, n > 0$[4] otherwise S_m or S_n will not be defined if any of x, y, or z is 0.

m and n cannot both be odd.[5] Otherwise, setting $(x, y, z) = (1, -1, 0)$ would give $S_{m+n} = 2$ while $S_m = S_n = 0$. By the same substitution, m and n cannot both be even, otherwise $S_{m+n} = S_m = S_n = 2$ so $\frac{2}{m+n} = \frac{2}{m} \cdot \frac{2}{n}$ from which $(m-2)(n-2) = 4$ giving $m = n = 4$, which fails because $\frac{S_8}{8} \neq \frac{S_4}{4} \cdot \frac{S_4}{4}$.

We must have m and n of opposite parity[6]. Suppose m is odd and n is even. If $n = 2$, setting $(x, y, z) = (-1, -1, 2)$ gives $(m-6) \cdot 2^m = -4m - 12$. This holds for $m = 3$ or $m = 5$, but for $m \geq 7$, there can be no solutions because the signs will be opposite.

We will show that if $n \geq 4$, it is impossible given the case $(x, y, z) = (-1, -1, 2)$. We have:

$$(1) \qquad (2^{m+n} - 2)(mn - m - n) = (m+n)(2^{m+1} - 2^{n+1} - 2)$$

Since $mn - m - n > 0$, each of the expressions in the parenthesis must be positive. For the last expression to be positive, we must have $m > n$, so that $m \geq 5$. Since $(m+n)(2^{m+1}-2^{n+1}-2) < (m+n)(2^{m+n}-2)$, it follows from (1) that $mn-m-n < m+n$, or that $(m-2)(n-2) < 4$. This is impossible since $m \geq 5$ and $n \geq 4$. *Q.E.D.*

[3]We should suspect that there are no other solutions because the indentity is simply too strange, and when mutiplied out does not look true at all.

[4]Simple cases should always come first in a proof.

[5]Breaking up a proof by contradiction based on odd and even is perhaps the most adaptable case-analysis.

[6]Parity refers to odd or even.