# TJ USAMO Practice 2

VMT Math Team

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1. (USAMO 1973) Let $X_n$ and $Y_n$ denote two sequences of integers defined as follows:

$$X_0 = 1, X_1 = 1, X_{n+1} = X_n + 2X_{n-1} \quad (n = 1,\ 2,\ 3,\ \ldots)$$
$$Y_0 = 1, Y_1 = 7, Y_{n+1} = 2Y_n + 3Y_{n-1} \quad (n = 1,\ 2,\ 3,\ \ldots)$$

   Prove that, except for $Y_0$, there is no $Y_i$ that appears in the sequence $X_j$.

2. (USAMO 1978) Nine mathematicians meet at an international conference and discover that among any three of them, at least two speak a common language. If each of the mathematicians speaks at most three languages, prove that there are at least three of the mathematicians who can speak the same language.

3. (USAMO 1979) Given three identical $n$-faced dice whose corresponding faces are indentically numbered with arbitrary integers, prove that if they are tossed at random, the probability that the sum of the bottom three face numbers is divisible by three is greater than or equal to $\frac{1}{4}$.

4. (USAMO 1981) If $x$ is a positive real number, and $n$ is a positive integer, prove that

$$[nx] \geq \frac{[x]}{1} + \frac{[2x]}{2} + \frac{[3x]}{3} + \cdots + \frac{[nx]}{n}$$

   where [t] denotes the greatest integer less than or equal to $t$.

5. (USAMO 1982) Let $S_r = x^r + y^r + z^r$ with $x, y, z \in \mathbb{R}$. It is known that if $S_1 = 0$,

$$\frac{S_{m+n}}{m+n} = \frac{S_m}{m}\frac{S_n}{n}$$

   For $(m, n) = (2, 3), (3, 2), (2, 5)$, or $(5, 2)$. Determine all other pairs of integers $(m, n)$, if any, so that the above identity holds for all real numbers $x, y, z$ such that $x+y+z = 0$.