TJ USAMO Practice 3 Solutions

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1. (USAMO 1979) Determine all non-negative integral solutions $(n_1, n_2, \ldots, n_{14})$ if any, apart from permutations, of the Diophantine equation

$$n_1^4 + n_2^4 + \cdots + n_{14}^4 = 1,599$$

Solution

There are no solutions in integers.

We take this equation modulo 16.[1] We get $\overbrace{(0,1) + (0,1) + \cdots + (0,1)}^{14} \equiv 15(\text{modulo } 16)$. Clearly, this has no solutions, and we are done.

2. Prove that, for $a, b, c > 0$,

$$a^3 + b^3 \geq a^2b + ab^2$$

and

$$\frac{1+a^2}{b} + \frac{1+b^2}{c} + \frac{1+c^2}{a} \geq 6$$

Solution

The first inequality can be demonstrate by some factoring: $a^3 + b^3 = (a+b)(a^2 - ab + b^2) \geq a^2b + ab^2 = (a+b)(ab)$. $a+b$ is positive, so the desired inequality is equivalent to $a^2 - ab + b^2 \geq ab \iff a^2 - 2ab + b^2 \geq 0 \iff (a-b)^2 \geq 0$, and this is true by the trivial inequality.

The second inequality can be solved by $AM - GM$. $\frac{1+a^2}{b} + \frac{1+b^2}{c} + \frac{1+c^2}{a} = \frac{1}{b} + \frac{a^2}{b} + \frac{1}{c} + \frac{b^2}{c} + \frac{1}{a} + \frac{c^2}{a} \geq 6\sqrt[6]{\frac{a^2b^2c^2}{bbccaa}} = 6$.

[1] Any time we are given a diophantine equation, modulos are involved, or should at least be tried. In general, if we want few residues n^k where n and k are integers, it is a good idea to try modulo p where $\phi(p) = k$, especially if p is prime. It is also a good idea to try modulo k^2. In this instance, we try modulo 4^2.

3. (MOP 2003) Let ABC be a triangle, and let H and ω be its orthocenter and circumcircle respectively. Let $\overline{BD}$ be a diameter of ω. Prove that $AHCD$ is a parallelogram.

   Solution

   Because $\overline{BD}$ is a diameter of ω, $\angle BAD$ and $\angle BCD$ are right angles, so $\overline{AB} \perp \overline{AD}$ and $\overline{BC} \perp \overline{CD}$. Let $E \in \overline{AB} : \overline{AB} \perp \overline{CE}$,[2] and define F analogously.[3]. Because C, H, and E are collinear, it follows that $\overline{CE} \parallel \overline{HC} \Longrightarrow \overline{HC} \perp \overline{AB} \perp AD \iff \overline{HC} \parallel \overline{AD}$. By a similar argument, $\overline{AH} \perp \overline{BC} \perp \overline{CD} \iff \overline{AH} \parallel \overline{CD}$. And we are done.

4. (MOP 2003) Find

   $$\sum_{k=1}^{n} k!(k^2+k+1)$$

   in closed form.

   Solution

   We apply telescoping to obtain the closed for equivalent:

   $$\begin{aligned}
   \sum_{k=1}^{n} k!(k^2+k+1) &= \sum_{k=1}^{n} (k+1)^2 \cdot k! - k \cdot k! && (1)\\
   &= \sum_{k=1}^{n} (k+1)(k+1)! - k \cdot k! && (2)\\
   &= (2 \cdot 2! - 1 \cdot 1!) + \cdots + ((n+1)\cdot(n+1)! - n \cdot n!) && (3)\\
   &= (n+1)\cdot(n+1)! - 1 && (4)
   \end{aligned}$$

5. (USAMO 1996) An ordered n-tuple

   $$(x_1, x_2, \ldots, x_n)$$

   in which each term is either 0 or 1 is called a *binary sequence of length n*. Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to 0, 1, 0 in that order. Let b_n be the number of binary sequences of length n that contain no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that $b_{n+1} = 2a_n$ for all positive integers n.

   Solution

   Let A, B be bit vectors of length n and $n+1$, respectively. Consider the mapping $A_{i-1} \longrightarrow B_i$ where $B_0 = 0$ or 1, $B_i = B_{i-1}$ if $A_{i-1} = 0$, $B_i = 1 - A_{i-1}$ if $A_{i-1} = 0$. Then 0011 and 1100 occur in B if and only if 010 occurs in A. This mapping has an inverse, where $A_i = 0$ if $B_i = B_{i+1}$ and 1 otherwise. The inverse mapping will create an initial digit and a sequence A. Therefore the number of sequences B is twice the

[2] $E \in \overline{AB} : \overline{AB} \perp \overline{CE}$ means that E is a point on line $\overline{AB}$ such that $\overline{AB}$ and $\overline{CE}$ are perpendicular.
[3] i.e. $F \in \overline{BC} : \overline{BC} \perp \overline{AF}$.

number of sequences A, because there is a one-to-one mapping between sequences B and sequences A with one of two additional digits. Since any sequence B fits the criteria for b_{n+1} and vice versa, and the same is true for A and a_n, we have that $b_{n+1} = 2a_{n+1}$, QED.